Chapter 8 POTENTIAL THEORY IN THREE DIMENSIONS

The theory of the preceding chapter, when generalized to three or more dimensions becomes considerably complicated. The development of this theory during the 19th century was motivated to a considerable extent by physical intuition. The study of fields of force and velocity of fluid flows led to the theorems on integration in severable variables which are in this chapter. More modern expositions of this material lean heavily on algebraic developments of the late 19th and early 20th centuries. Although the mathematics has significantly improved with the introduction of the notions of differential forms and invariance, the intuition provided by concrete interpretations has been lost. We shall lean heavily on the interpretation by fluid flows, thereby sacrificing some mathematical rigor for a little bit of concreteness. We certainly should point out that the importance of the subject of differential forms by far transcends its use in putting the divergence theorem on firm ground. This theory has had major impact on all branches of modern research mathematics and physics. We have however selected to complete our story rather than begin to suggest a new one.

A **fluid flow** is given by a function $\boldsymbol{\phi}(\mathbf{x}_0, t)$ defined for $\mathbf{x}_0$ in some domain D in R^3 and t on an interval in R about the origin. We require that

(i) $\boldsymbol{\phi}$ is continuously differentiable in all variables,
(ii) $\boldsymbol{\phi}(\mathbf{x}_0, 0) = \mathbf{x}_0$, all $\mathbf{x}_0 \in D$,
(iii) for fixed t, the transformation $\mathbf{x}_0 \to \boldsymbol{\phi}(\mathbf{x}_0, t)$ is one-to-one and has a nonsingular differential.

The value $\boldsymbol{\phi}(\mathbf{x}_0, t)$ represents the space position at time t of the particle which was at $\mathbf{x}_0$ at time $t = 0$. We shall refer to $\mathbf{x}_0$ as the **particle coordinate** and to $\mathbf{x} = \boldsymbol{\phi}(\mathbf{x}_0, t)$ as the **space coordinate**. Condition (ii) asserts that the particle and space coordinates coincide at $t = 0$. Condition (iii) asserts that the relation between particle and space coordinates at any time t is invertible: we can recapture the initial position of a particle from its position at any time. We shall denote the inverse of $\boldsymbol{\phi}$ by $\boldsymbol{\psi}$: $\mathbf{x} = \boldsymbol{\phi}(\mathbf{x}_0, t)$ if and only if $\mathbf{x}_0 = \boldsymbol{\psi}(\mathbf{x}, t)$.

The curve given by $\mathbf{x} = \phi(\mathbf{x}_0, t)$ is the **path of motion** of the particle $\mathbf{x}_0$. The velocity of $\mathbf{x}_0$ at time t is, of course, $(\partial\boldsymbol{\phi}/\partial t)(\mathbf{x}_0, t)$. If we fix the time t, the collection of velocity vectors forms a field, denoted by $\mathbf{v}(\mathbf{x}, t)$ (referring of course to spatial coordinates) called the **velocity field of the flow.** $\mathbf{v}(\mathbf{x}, t)$ is the velocity of the particle at $\mathbf{x}$ at time t. We have already noted that

$$\mathbf{v}(\mathbf{x}, t) = \left.\frac{\partial\boldsymbol{\phi}(\mathbf{x}_0, t)}{\partial t}\right|_{\mathbf{x}_0 = \boldsymbol{\psi}(\mathbf{x}, t)} \tag{8.1}$$

If the velocity field is independent of time, we say that the flow is **steady.**

The velocity field of a flow completely determines the flow: the path of motion $\mathbf{x} = \mathbf{u}(t)$ of a particle $\mathbf{x}_0$ is the solution of the differential equation

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} &= \mathbf{v}(\mathbf{u}, t) \\ \mathbf{u}(0) &= \mathbf{x}_0 \end{aligned} \tag{8.2}$$

By (8.1) the solution is given by $\mathbf{u}(t) = \boldsymbol{\phi}(\mathbf{x}_0, t)$, for (8.1) can be rewritten as

$$\mathbf{v}(\boldsymbol{\phi}(\mathbf{x}_0, t), t) = \frac{\partial\boldsymbol{\phi}(\mathbf{x}_0, t)}{\partial t}$$

Thus the equation of flow is recaptured from the velocity field by solving Equation (8.2).

This introduction recapitulates what we have already learned about fluid flows. In the subsequent section we shall develop the mathematics required to study the evolution through time of a given mass of fluid. We shall see that the various laws of conservation of physics (mass, energy) correspond to mathematical theorems (divergence theorem, Stokes' theorem).

8.1 Divergence and the Equation of Continuity

Let us begin with a fluid flowing through a domain in R^3 according to the equation $\mathbf{x} = \boldsymbol{\phi}(\mathbf{x}_0, t)$. According to reasonable physical assumptions, if we define the density at a point $\mathbf{p}$ as the limit

$$\rho(\mathbf{p}) = \lim_{\Delta \to \mathbf{p}} \frac{\text{mass } \Delta}{\text{vol } \Delta}$$

as the domain Δ shrinks uniformly down to $\mathbf{p}$, then the mass of any domain is given by integration of the density function ρ. In our case, that of a fluid in motion, we shall express the density of the fluid at the point $\mathbf{x}$ at time t as $\rho(\mathbf{x}, t)$. Thus, for any domain D, the mass of fluid in D at time t is

$$\int_D \rho(\mathbf{x}, t)\, dV$$

We can also consider the density at a particle: $\rho(\boldsymbol{\phi}(\mathbf{x}_0, t), t)$ is the density of the fluid at time t at the particle (originally at) $\mathbf{x}_0$. (More generally, we always have this option of referring measurable quantities to either the spatial, or the particle coordinates. This option is a source of some confusion, as well as deepening, of our understanding.)

The law of conservation of matter asserts that the mass of a given object is independent of time. If we fix a domain D, the space occupied at time t by the fluid originally in D is the domain $D_t = \{\boldsymbol{\phi}(\mathbf{x}_0, t) : \mathbf{x}_0 \in D\}$. The mass of fluid in D_t is

$$\int_{D_t} \rho(\mathbf{x}, t)\, dV$$

Since mass must be conserved, this must be independent of t. Thus the law of conservation of mass can be expressed by this equation:

$$\frac{\partial}{\partial t} \int_{D_t} \rho(\mathbf{x}, t)\, dV = 0 \tag{8.3}$$

for any domain D. We would prefer to state this as an equation involving functions of points, rather than domains. In order to do that we must know how to carry through the differentiation implied in (8.3). The problem with

(8.3) is that we have a variable domain of integration. This can be solved by replacing that integral by one over D. We shall now briefly interrupt this discussion with a description of the formula for change of variables in an integral. This will allow us to compute (8.3).

Suppose now that we are given a one-to-one transformation $\mathbf{y} = \mathbf{F}(\mathbf{x})$ of a domain D onto a domain Δ. We assume that $\mathbf{F}$ is continuously differentiable, and its differential is everywhere nonsingular. We shall require also that $d\mathbf{F}(\mathbf{x})$ is orientation-preserving: that is, that it maps the standard basis $\mathbf{E}_1 \to \mathbf{E}_2 \to \mathbf{E}_3$ into a right-handed system. Writing $\mathbf{x} = (x^1, x^2, x^3)$, $\mathbf{y} = (y^1, y^2, y^3)$, the image of $\mathbf{E}_i$ under the linear transformation $d\mathbf{F}(\mathbf{x})$ is just $(\partial\mathbf{F}/\partial x^i)(\mathbf{x})$. Thus we require that

$$\frac{\partial \mathbf{F}}{\partial x^1}(\mathbf{x}) \to \frac{\partial \mathbf{F}}{\partial x^2}(\mathbf{x}) \to \frac{\partial \mathbf{F}}{\partial x^3}(\mathbf{x})$$

be a right-handed system, which is the same as asking that

$$\det \frac{\partial(y^1, y^2, y^3)}{\partial(x^1, x^2, x^3)}(\mathbf{x}) = \det\left(\frac{\partial \mathbf{F}}{\partial x^1}(\mathbf{x}), \frac{\partial \mathbf{F}}{\partial x^2}(\mathbf{x}), \frac{\partial \mathbf{F}}{\partial x^3}(\mathbf{x})\right) > 0$$

With these hypotheses we have the following formula for integration under the change of variable $\mathbf{F}$. If f is an integrable function on Δ, then

$$\int_\Delta f(\mathbf{y})\, dV = \int_\Delta f(\mathbf{F}(\mathbf{x})) \det \frac{\partial(y^1, y^2, y^3)}{\partial(x^1, x^2, x^3)}\, dV \tag{8.4}$$

We shall defer the derivation of this formula to the end of this section. The motivating idea is that it is true in the small: if the function f is constant, and the transformation $\mathbf{F}$ is a linear transformation, and D is a rectangle, then (8.3) just says that the volume of the parallelepiped $\mathbf{F}(D)$ is $\det \mathbf{F} \cdot \mathrm{vol}(D)$ (an easily verified fact). The general case follows by locally approximating by this case and summing over the whole domain.

Examples

1. Find $\int_B x^2 y^4\, dV$, where B is the unit ball. We use spherical coordinates for this computation:

$$x = r \sin\theta \cos\phi \qquad y = r \sin\theta \sin\phi \qquad z = r\cos\theta$$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}$$

so

$$\det \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin \theta$$

$$\begin{aligned}\int_B x^2 y^4 \, dV &= \int_0^1 \int_{-\pi}^{\pi} \int_0^{\pi} r^8 \sin^6 \theta \cos^2 \phi \sin^5 \phi \, dr \, d\phi \, d\theta \\ &= \int_0^1 r^8 \, dr \cdot \int_{-\pi}^{\pi} \cos^2 \phi \sin^5 \phi \, d\phi \cdot \int_0^{\pi} \sin^6 \theta \, d\theta \\ &= \frac{1}{9} \cdot \frac{16}{105} \cdot \frac{7\pi}{16} = \frac{7\pi}{945}\end{aligned}$$

2. $\int_D (x^2 - y^2) \, dx \, dy$, where $D = \{0 \le x \le 1, x - 1 \le y \le x\}$ becomes

$$\frac{1}{2} \int_0^1 \int_0^1 uv \, du \, dv = \frac{1}{8}$$

under the change of variable $u = x - y$, $v = x + y$.

3. $\int_B (x^2 + y^2 + z^2) \, dx \, dy \, dz$, where B is the domain

$$B = \{x^2 + y^2 \le 1, 0 \le z \le 2\}$$

This can be easily computed in cylindrical coordinates:

$$\begin{aligned}\int_B (x^2 + y^2 + z^2) \, dx \, dy \, dz &= \int_0^{2\pi} \int_0^1 \int_0^2 (r^2 + z^2) r \, d\theta \, dr \, dz \\ &= 2\pi \int_0^1 \left(r^3 + \frac{8}{3} r\right) dr \\ &= \frac{19\pi}{6}\end{aligned}$$

We return to our fluid flow given by $\mathbf{x} = \boldsymbol{\phi}(\mathbf{x}_0, t)$. We shall express it, for the sake of compution, in coordinates:

$$(x^1, x^2, x^3) = \boldsymbol{\phi}(x_0{}^1, x_0{}^2, x_0{}^3, t) \tag{8.5}$$

Since (8.5) reduces to the identity for $t = 0$, we have

$$\left.\frac{\partial(x^1, x^2, x^3)}{\partial(x_0{}^1, x_0{}^2, x_0{}^3)}\right|_{t=0} = \mathbf{I} \tag{8.6}$$

Thus, the determinant

$$J_t(\mathbf{x}_0) = \det \frac{\partial(x^1, x^2, x^3)}{\partial(x_0{}^1, x_0{}^2, x_0{}^3)}$$

is positive for all small t, so we can apply the change of variable formula to the computation of (8.3) for fixed small t. We now have the mass conservation law expressed by

$$0 = \frac{\partial}{\partial t}\int_{D_t} \rho(\mathbf{x}, t)\, dV = \frac{\partial}{\partial t}\int_D \rho(\boldsymbol{\phi}(\mathbf{x}_0, t), t) J_t\, dV = \int_D \frac{\partial}{\partial t}(\rho J_t)\, dV$$

(The final equation follows since differentiation under the integral is now allowable.) Since this must be true for every domain D, the integrand is identically zero:

$$\frac{\partial}{\partial t}(\rho J_t) = 0 \tag{8.7}$$

We can explicitly compute that derivative for $t = 0$, using (8.6). First, let us consider

$$\left.\frac{\partial}{\partial t} J_t\right|_{t=0} = \frac{\partial}{\partial t} \det \left.\frac{\partial(x^1, x^2, x^3)}{\partial(x_0{}^1, x_0{}^2, x_0{}^3)}\right|_{t=0} \tag{8.8}$$

The determinant is the usual sum of products of the various partial derivatives $\partial x^i/\partial x_0{}^j$. The derivative of such a product will have three terms; in each one of which only one term is differentiated with respect to t. Each term is of the form

$$\left.\frac{\partial}{\partial t}\left(\frac{\partial r^1}{\partial s^1}\right)\right|_{t=0} \cdot \left.\frac{\partial r^2}{\partial s^2}\right|_{t=0} \cdot \left.\frac{\partial r^3}{\partial s^3}\right|_{t=0} \tag{8.9}$$

where $\{r^1, r^2, r^3\}$ is a permutation of $\{x^1, x^2, x^3\}$, and $\{s^1, s^2, s^3\}$ a permutation of $\{x_0^1, x_0^2, x_0^3\}$. According to (8.6)

$$\left.\frac{\partial r}{\partial s}\right|_{t=0} = 0 \text{ if } s \neq r_0 \qquad \left.\frac{\partial r}{\partial s}\right|_{t=0} = 1 \text{ if } s = r_0$$

Thus the only relevant terms (8.9) are those where $s^2 = r_0^2$, $s^3 = r_0^3$ and, a fortiori, $s^1 = r_0^1$. Finally, by the equality of mixed partial derivatives,

$$\left.\frac{\partial}{\partial t}\left(\frac{\partial x^i}{\partial x_0^i}\right)\right|_{t=0} = \left.\frac{\partial}{\partial x_0^i}\left(\frac{\partial x^i}{\partial t}\right)\right|_{t=0} = \left.\frac{\partial v^i}{\partial x_0^i}\right|_{t=0}$$

where $\mathbf{v} = (v^1, v^2, v^3)$ is the velocity field of the flow (recall Equation (8.1)). Thus, the computation of (8.8) is complete: there are only three relevant terms, for $r^1 = x^1, x^2, x^3$, respectively, and we have

$$\left.\frac{\partial}{\partial t} J_t\right|_{t=0} = \left.\frac{\partial v^1}{\partial x_0^1} + \frac{\partial v^2}{\partial x_0^2} + \frac{\partial v^3}{\partial x_0^3}\right|_{t=0} \tag{8.10}$$

Definition 1. Let $\mathbf{v} = (v^1, v^2, v^3)$ be a differentiable vector field defined in a domain in R^3. The **divergence** of $\mathbf{v}$ is the function defined by

$$\operatorname{div} \mathbf{v} = \sum_{i=1} \frac{\partial v^i}{\partial x^i}$$

The name will appear presently to be justified. We now summarize our discussion in the following assertion.

Proposition 1. (Equation of Continuity) *Let $\mathbf{v}(\mathbf{x}, t)$ be the velocity field of a fluid flow, and $\rho(\mathbf{x}, t)$ its density. The law of mass conservation takes this form:*

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = \frac{\partial \rho}{\partial t} + \sum_{i=1}^{3} v_i \frac{\partial \rho}{\partial x^i} + \rho \operatorname{div} \mathbf{v} = 0 \tag{8.11}$$

Proof. Referring to the preceding discussion we have seen from (8.7) that the law of mass conservation asserts that

$$\frac{\partial}{\partial t}\left(\rho(\boldsymbol{\phi}(\mathbf{x}_0, t), t)\, J_t(\mathbf{x}_0)\right) = 0$$

for all t, $\mathbf{x}_0$. Evaluating at $t = 0$, this becomes

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\rho(\boldsymbol{\phi}(\mathbf{x}_0, t), t))|_{t=0} \cdot J_0(\mathbf{x}_0) + \rho(\boldsymbol{\phi}(\mathbf{x}_0, t), t)) \frac{\partial}{\partial t} J_t(\mathbf{x}_0)|_{t=0} \\
&\quad = \sum_{i=1}^{3} \frac{\partial \rho}{\partial x^i}(\mathbf{x}_0, 0) \frac{\partial x^i}{\partial t}(\mathbf{x}_0, 0) + \frac{\partial \rho}{\partial t}(\mathbf{x}_0, 0) + \rho(\mathbf{x}_0, 0) \operatorname{div} \mathbf{v}(\mathbf{x}_0, 0)
\end{aligned} \tag{8.12}
$$

The second expression follows from our computation above terminating in (8.10), and the fact that $J_0(\mathbf{x}_0) = 1$, $\mathbf{x}_0 = \boldsymbol{\phi}(\mathbf{x}_0, 0)$. Now, we could have started our clock at any time; there is nothing special about the time $t = 0$ except that our formulas are most easily computed there. Thus, (8.12) must hold for all $(\mathbf{x}, t)$ since it is valid for all $(\mathbf{x}_0, 0)$. Thus (8.11) is true. We leave the first equality as an exercise.

Equation (8.11) can be referred to the particle coordinates of the motion:

$$
\begin{aligned}
&\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{x}=\boldsymbol{\phi}(\mathbf{x}_0, t)} + \sum_{i=1}^{3} \frac{\partial x^i}{\partial t}(\mathbf{x}_0, t) \left.\frac{\partial \rho}{\partial x^i}\right|_{\mathbf{x}=\boldsymbol{\phi}(\mathbf{x}_0, t)} \\
&\qquad + \rho(\boldsymbol{\phi}(\mathbf{x}_0, t), t) \operatorname{div} \frac{\partial \mathbf{x}}{\partial t}(\mathbf{x}_0, t) = 0
\end{aligned}
$$

which compresses into

$$
\frac{\partial}{\partial t}\rho(\boldsymbol{\phi}(\mathbf{x}_0, t), t) + \rho(\boldsymbol{\phi}(\mathbf{x}_0, t), t) \operatorname{div} \frac{\partial \mathbf{x}}{\partial t}(\mathbf{x}_0, t) = 0 \tag{8.13}
$$

This relates the time rate of change of density at a particle with the rate of change of its position. A fluid flow is called **incompressible** if the same mass always occupies the same volume. For an incompressible fluid flow we must therefore have that $\int_{D_t} dV$ is constant for any initial domain D. Thus

$$
0 = \frac{\partial}{\partial t}\int_{D_t} dV = \frac{\partial}{\partial t}\int_D J_t \, dV = \int_D \frac{\partial \mathbf{x}_t}{\partial t^t} dV = \int_D \operatorname{div} \mathbf{v} \, dV \tag{8.14}
$$

for every domain D. Thus $\operatorname{div} \mathbf{v} = 0$ is the necessary and sufficient condition for a flow to be incompressible. By the equation of continuity (in the form (8.13)) this is the same as asking that the density at a particle is also independent of time.

Corollary 1. *$\mathbf{v}$ is the velocity of flow of an incompressible fluid if and only if* $\operatorname{div} \mathbf{v} = 0$.

Corollary 2. *The fluid is incompressible if and only if the density at a particle is constant under all flows of the fluid.*

Now the integral $\int_D \operatorname{div} \mathbf{v}\, dV$ is the rate of expansion of the fluid in D, according to our computation (8.14). (Hence, the name divergence.) We could also calculate the "infinitesimal expansion" of D by calculating the amount of fluid which enters during an "infinitesimal" amount of time, and subtracting from it the amount of fluid that leaves. The mathematical expression of this will be an integral over the boundary of the domain D. The fact that this is the same as $\int_D \operatorname{div} \mathbf{v}\, dV$ is the divergence theorem, which is a fundamental fact in calculus. We shall return to this theorem and its implications in Section 8.5.

Examples

4. Consider the flow given by the equations

$$x = x_0(1 + t) + ty_0 \qquad y = y_0(1 - t) + tx_0 \qquad z = z_0 e^t$$

If D is the original position of a mass of fluid,

$$D_t = \{(x_0(1 + t) + ty_0, y_0(1 - t) + tx_0, z_0 e^t); (x_0, y_0, z_0) \in D\}$$

and the volume of D_t is

$$\int_{D_t} dV = \int_D \det \frac{\partial(x, y, z)}{\partial(x_0, y_0, z_0)}\, dV$$

$$= \int_D e^t(1 - 2t^2)\, dV = e^t(1 - 2t^2) \operatorname{vol}(D)$$

Since $\dfrac{\partial}{\partial t} \operatorname{vol}(D_t) = \int_D \operatorname{div} \mathbf{v}\, dV$ for every domain D, we have

$$\operatorname{div} \mathbf{v}(\mathbf{x}, t) = \frac{\partial}{\partial t} e^t(1 - 2t^2) = e^t(1 - 4t - 2t^2)$$

5. For this flow:

$$x = x_0 e^t \qquad y = y_0 e^{-t} \qquad z = z_0 e^t + x_0(1 - e^t)$$

we have

$$\frac{\partial \mathbf{x}}{\partial t} = (x_0 e^t, \ -y_0 e^{-t}, z_0 e^t - x_0 e^t)$$

so $\mathbf{v}(\mathbf{x}, t) = (x, \ -y, z - xe^{-t})$ and $\operatorname{div} \mathbf{v} = 1$. Thus, for any domain D, $(\partial/\partial t) \operatorname{vol}(D_t) = 1 \cdot \operatorname{vol}(D_t)$, so $\operatorname{vol}(D_t) = e^t \operatorname{vol}(D)$. If $\rho(\mathbf{x}, t)$ is the density function at time t, the equation of continuity allows us to find ρ in terms of its initial values. Let $\rho(\mathbf{x}_0, 0) = \rho(\mathbf{x}_0)$ be given. Then, according to (8.13), if $\tilde{\rho}(\mathbf{x}_0, t)$ is the particle density, we have

$$\frac{\partial \tilde{\rho}}{\partial t} + \tilde{\rho} \cdot 1 = 0$$

$$\tilde{\rho}(\mathbf{x}_0, 0) = \rho(\mathbf{x}_0)$$

Thus

$$\tilde{\rho}(\mathbf{x}_0, t) = \rho(\mathbf{x}_0)e^{-t} \qquad \rho(\mathbf{x}, t) = e^{-t}\rho(xe^{-t}, ye^t, z - x(e^{-t} - 1)e^{-t})$$

6. Suppose an incompressible fluid flows steadily in the direction $\mathbf{a} = (a^1, a^2, a^3)$. That is, the path lines are parallel to the vector $\mathbf{a}$. Then the speed is constant along the paths. For the velocity field is

$$\mathbf{v}(\mathbf{x}, 0) = \phi(\mathbf{x})\mathbf{a}$$

where ϕ is a scalar function (the speed), and since $\mathbf{v}$ is divergence free, we have

$$\operatorname{div} \mathbf{v} = \frac{\partial \phi}{\partial x^1} a^1 + \frac{\partial \phi}{\partial x^2} a^2 + \frac{\partial \phi}{\partial x^3} a^3 = 0$$

But then

$$d\phi(\mathbf{x})(\mathbf{a}) = \langle \nabla \phi(\mathbf{x}), \mathbf{a} \rangle = 0$$

for all $\mathbf{x}$, so ϕ is constant along the lines parallel to $\mathbf{a}$; but these are the paths of motion.

Integration Under a Coordinate Change

Theorem 8.1. *Let $(u, v, w) = \mathbf{F}(x, y, z)$ be an orientation-preserving change of coordinates valid in the domain D in x, y, z space. Let $\Delta = \{\mathbf{F}(x, y, z) : (x, y, z) \in D\}$. If g is a function continuous on D, then*

$$\int_D g(x, y, z)\, dx\, dy\, dz = \int_\Delta g(\mathbf{F}^{-1}(u, v, w) \det \frac{\partial(x, v, z)}{\partial(u, v, w)}\, du\, dv\, dw$$

Proof. The proof consists in a series of reductions terminating in the one-variable case. It is enough to show that for any point $\mathbf{p} \in D$, this theorem is true for some rectangle centered at $\mathbf{p}$. For, once this is shown, we may cover D by finitely many such rectangles $R_1, \ldots, R_n$. If $\{\rho_1, \ldots, \rho_n\}$ is a partition of unity subordinate to $\{R_1, \ldots, R_n\}$, then $\rho_i \cdot g$ is zero outside R_i. The theorem is thus true for each $\rho_i \cdot g$. Summing over i, we obtain the general result.

Thus we may concentrate our attention on a particular point $\mathbf{p}_0$ in D, which we take to be the origin. If the theorem is valid for the coordinate changes $\mathbf{u} = \mathbf{F}(\mathbf{x})$, $\mathbf{y} = \mathbf{G}(\mathbf{u})$, then it is also true for the composed mapping $\mathbf{y} = \mathbf{G}(\mathbf{F}(\mathbf{x}))$, simply because

$$\frac{\partial(x^1, x^2, x^3)}{\partial(y^1, y^2, y^3)} = \frac{\partial(x^1, x^2, x^3)}{\partial(u^1, u^2, u^3)} \cdot \frac{\partial(u^1, u^2, u^3)}{(\partial y^1, y^2, y^3)}$$

We will decompose our mapping into a composition of four special cases, for each of which the theorem is easy. The general result will follow by composing these mappings.

First of all, let $\mathbf{T}$ be the linear mapping

$$\mathbf{T}(\mathbf{x}) = \left.\frac{\partial(x, y, z)}{\partial(u, v, w)}\right|_{(u, v, w) = 0} \cdot (\mathbf{x})$$

Then $\mathbf{F} = (\mathbf{F} \circ \mathbf{T}) \circ \mathbf{T}^{-1}$ and $\mathbf{F} \circ \mathbf{T}$ has the property that its Jacobian at $\mathbf{0}$ is the identity. The theorem is easily seen to be true for a linear mapping (Problem 5), so we need only prove it for $\mathbf{F} \circ \mathbf{T}$.

Our situation is now this: we are given a change of coordinates $(u, v, w) = \mathbf{G}(x, y, z)$ defined at the origin such that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)}(\mathbf{0}) = \mathbf{I}$$

It follows that

$$\frac{\partial(u, y, z)}{\partial(x, y, z)}(\mathbf{0}) = \mathbf{I}$$

$$\frac{\partial(u, v, z)}{\partial(x, y, z)}(\mathbf{0}) = \mathbf{I}$$

Thus, by the inverse mapping theorem there is a neighborhood B of $\mathbf{0}$ in which $(x, y, z), (u, y, z), (u, v, z), (u, v, w)$ are all bona fide orientation-preserving coordinate systems. If we denote the respective coordinate changes as follow

$$\begin{aligned}
\mathbf{F}_1(x, y, z) &= (u, y, z)\\
\mathbf{F}_2(u, y, z) &= (u, v, z)\\
\mathbf{F}_3(u, v, z) &= (u, v, w)
\end{aligned}$$

then $\mathbf{F} = \mathbf{F}_3 \circ \mathbf{F}_2 \circ \mathbf{F}_1$. Each $\mathbf{F}_i$ changes only one coordinate at a time, and we need only to prove the theorem for each $\mathbf{F}_i$. Since the proof of each case is the same, we shall do it only once.

Now, here we do our computation. Let

$$\begin{aligned}
u &= h(x, y, z)\\
v &= y\\
w &= z
\end{aligned}$$

be a coordinate change defined on a rectangle

$$R = \{-a \le x \le a,\ -b \le y \le b,\ -c \le z \le c\}$$

centered at the origin. Let

$$\Delta = \{(u, v, w) \colon u = h(x, v, w),\ -a \le x \le a,\ -b \le v \le b,\ -c \le w \le c\}$$

If now g is a continuous function on R,

$$\int_R g(x, y, z)\, dx\, dy\, dz = \int_{-b}^{b} \int_{-c}^{c} \left[\int_{-a}^{a} g(x, y, z)\, dx \right] dy\, dz \tag{8.15}$$

Now, according to the theorem of change of variable in one dimension

$$\int_{-a}^{a} g(x, y, z)\, dx = \int_{h(-a,y,z)}^{h(a,y,z)} g(h^{-1}(x, y, z, v, w))\, \frac{\partial h^{-1}}{\partial u}(u, y, z)\, du$$

Thus (8.15) becomes

$$\int_{-b}^{b} \int_{-c}^{c} \left[\int_{h(-a,v,w)}^{h(a,v,w)} g(h^{-1}(u, v, w, v, w))\, \frac{\partial h^{-1}}{\partial u}(u, v, w)\, du \right] dv\, dw$$

$$= \int_\Delta g(h^{-1}(u, v, w, v, w)) \det \frac{\partial(x, y, z)}{(u, v, w)}\, du\, dv\, dw$$

The last equation follows from

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{pmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so

$$\det\left(\frac{\partial(x, y, z)}{\partial(u, v, w)}\right) = \det\left(\frac{\partial(u, v, w)}{\partial(x, y, z)}\right)^{-1} = \left(\frac{\partial h}{\partial x}\right)^{-1} = \frac{\partial h^{-1}}{\partial u}$$

• EXERCISES

1. Compute the area of these domains, using either spherical or cylindrical coordinates:
   (a) $x^2 + y^2 + z^2 \geq xyz$
   (b) $1 \geq x^2 + y^2 - z^2 \geq 0$
   (c) $x^2 + y^2 \leq z \leq 1$
   (d) $a^2x^2 + b^2y^2 + c^2z^2 \leq 1$
2. Integrate f over the domain D:
   (a) $f(x) = x^2y^2z^4 \qquad D = \{x^2 + y^2 + z^2 \leq 1\}$
   (b) $f(x) = xyz \qquad D = \{x^2 + y^2 \leq 1 \qquad 0 \leq z \leq 1\}$
   (c) $f(x) = x^2 + y^2 - z^2 \qquad D = \{a^2x^2 + b^2y^2 \leq 1 \qquad 0 \leq z \leq x^2 + y^2\}$
   (d) $f(x) = r \sin^2 \theta \cos^2 \phi \qquad D = \{0 \leq x(x^2 + y^2 + z^2) \leq 1\}$
3. What is the mass of a parabolic section:

$$0 \leq z \leq a(x^2 + y^2)$$

whose density is proportional to the distance from the xy plane?

4. Find the mass of the ball of radius 1, whose density is $\rho(x) = (1 + r)^{-1}$.
5. Let

$$x = x_0 + ty_0 \qquad y = y_0 e^t - tz_0 \qquad z = z_0 e^{-t} + tx_0$$

be the equations of a flow in space.
   (a) Compute the velocity field $\mathbf{v}(\mathbf{x}, t)$.
   (b) Compute the divergence of the flow.
   (c) Assuming an initial density function which is constant, find the density function $\rho(\mathbf{x}, t)$.
   (d) What is the mass of the fluid in the unit cube at time $t = 1$?
6. Which of these fluid flows is incompressible?
   (a) $\mathbf{v}(\mathbf{x}, t) = (-z, x, y)$
   (b) $\mathbf{v}(\mathbf{x}, t) = (z^2 - x^2, z - y, z)$
   (c) $x = x_0 e^t + (1 - t)y_0, \; y = y_0 e^{-t/2} + (1 - t)z_0, \; z = e^{-t/2}z_0$
   (d) $x = x_0 \cos t + y_0 \sin t \qquad y = y_0 \cos t - x_0 \sin t \qquad z = az_0(1 + t)$
   (e) $\mathbf{v}(\mathbf{x}, t) = (x \cos t, xy \sin t, ze^t)$

7. Find the volume at time $t = 1$ of the mass of fluid originally in the unit sphere under these flows:

(a) Exercise 6(a). (b) Exercise 6(c). (c) Exercise 6(d).

8. Show that a C^2 function in R^3 is harmonic if and only if it is the potential of the vector field of an incompressible flow. (*Hint*: div $\nabla f = \Delta f$.)

• PROBLEMS

1. A **radial field** is a field of the form

$$\mathbf{v}(\mathbf{x}) = \phi(\|\mathbf{x}\|)\mathbf{x}$$

Find all incompressible radial fields.

2. If L is a line in R^3, a flow around the axis L is one whose velocity field at any point is tangent to the cylinder with central line L. Show that the flow of Exercise 6(d) is a flow around the z axis. Find another such flow which is incompressible.

3. Find the incompressible flow whose path lines are the curves

$$x = x_0 + u \qquad y = y_0 + \sin u \qquad z = z_0$$

4. Find the incompressible flow whose path lines are the curves (in cylindrical coordinates)

$$z = Cr^{-1} \qquad \theta = \theta_0$$

(see Figure 8.1).

5. Prove Theorem 1 for the coordinate change $\mathbf{u} = \mathbf{T}(\mathbf{x})$, where $\mathbf{T}$ is a nonsingular linear transformation.

6. In the proof of Theorem 1, a function

$$u = h(x, y, z)$$

was found. It was tacitly assumed that $\partial h/\partial x > 0$. Why is that so? Express $\partial h^{-1}/\partial u$ in terms of the original functions $(u, v, w) = \mathbf{G}(x, y, z)$.

8.2 Curl and Rotation

The divergence of the velocity field of a flow measures the rate of expansion of the fluid in flow as we have seen. We shall now compute an indicator of its rotation around a given axis. Suppose

$$\mathbf{x} = \boldsymbol{\phi}(\mathbf{x}_0, t) \tag{8.16}$$

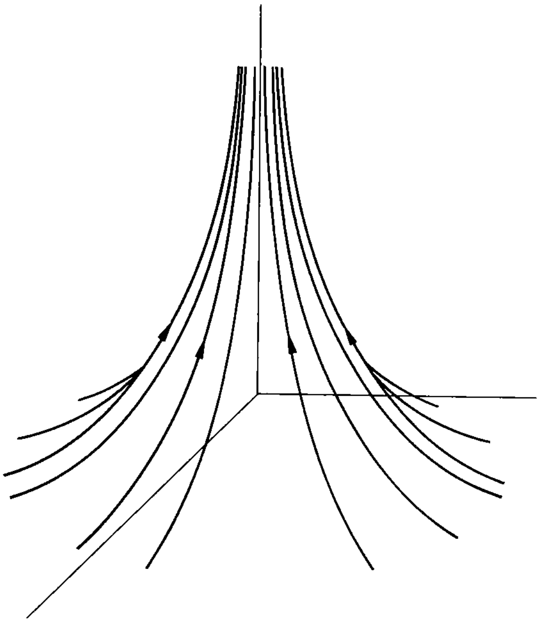

Figure 8.1

is the equation of motion of the flow. Let $\mathbf{x}_0$ be any point, and $\mathbf{n}$ a direction (unit) vector at the point $\mathbf{x}_0$. We shall compute the average angular velocity in the plane orthogonal to $\mathbf{n}$ at the point $\mathbf{x}_0$ in terms of the velocity field $\mathbf{v}$. We take $\mathbf{x} = \mathbf{0}$ for convenience. Since we are interested in the motion around the axis $\mathbf{n}$, relative to the motion of $\mathbf{0}$, we must work in coordinates relative to $\mathbf{0}$. What is the same, we shall subtract from the above motion a motion of translation by the image of $\mathbf{0}$, so that $\mathbf{0}$ remains fixed. Since translation involves no rotation, our computation will be valid for the original motion. Thus we replace (8.16) by the flow

$$\mathbf{x} = \boldsymbol{\psi}(\mathbf{x}_0, t) = \boldsymbol{\phi}(\mathbf{x}_0, t) = \boldsymbol{\phi}(\mathbf{0}, t) \tag{8.17}$$

so that in our new motion the origin is fixed.

Let C_r be a circle of radius r centered at $\mathbf{0}$ lying in the plane $\Pi(\mathbf{n})$ orthogonal to $\mathbf{n}$. Let $\mathbf{a}$ be a point on C_r. After a time t, the particle originally at $\mathbf{a}$ has moved to $\boldsymbol{\psi}(\mathbf{a}, t)$. Let $\mathbf{L}$ be the projection of $\boldsymbol{\psi}(\mathbf{a}, t) - \mathbf{a}$ onto the line tangent to C_r at $\mathbf{a}$ (see Figure 8.2). Let $\theta(t)$ be the angle at $\mathbf{0}$ in $\Pi(\mathbf{n})$ between $\mathbf{a}$ and $\mathbf{a} + \mathbf{L}$. Thus $\theta(t)$ is the angle in the plane orthogonal to $\mathbf{n}$ through which $\mathbf{a}$

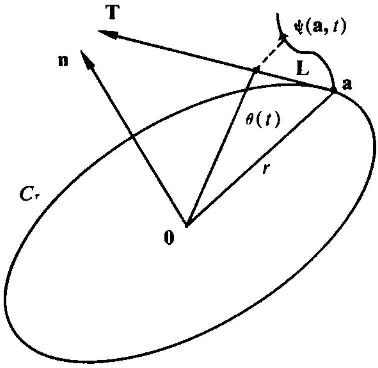

Figure 8.2

has moved (relative to $\mathbf{0}$) during the time t. Thus

$$\theta(t) = \sin^{-1}\frac{L}{r} = \sin^{-1}\frac{\langle \psi(\mathbf{a}, t) - \psi(\mathbf{a}), \mathbf{T}\rangle}{r}$$

when $\mathbf{T}$ is the unit tangent vector to C_r at $\mathbf{a}$. Dividing by t and letting $t \to 0$, we obtain the **angular velocity** for the particle $\mathbf{a}$ in $\Pi(\mathbf{n})$ as

$$\left(\sin^{-1}\frac{L}{r}\right)'\Bigg|_{t=0} = \frac{L'}{(1-(L/r^2)^{1/2}}\Bigg|_{t=0} = \left\langle \frac{\partial \psi}{\partial t}(\mathbf{a}, 0), \mathbf{T}\right\rangle$$
$$= \langle \mathbf{v}(\mathbf{a}, 0) - \mathbf{v}(\mathbf{0}, 0), \mathbf{T}\rangle$$

according to (8.17). The sum over all of C_r of this angular velocity is called the **total circulation of the flow about** C_r and is denoted circ(C_r). Thus

$$\text{circ}(C_r) = \int_{C_r} \langle \mathbf{v}(\mathbf{a}, 0) - \mathbf{v}(\mathbf{0}, 0), \mathbf{T}\rangle\, ds \tag{8.18}$$

This number, calculated for small r gives us some idea of the instantaneous rotation of the flow around $\mathbf{n}$ at $\mathbf{0}$. If we suitably normalize ((8.17) tends to zero as fast as $r \to 0$), and take the limit as $r \to 0$ we will have the same kind of information, but it will be given by a point function, rather than a function of circles.

Definition 2. Let $\mathbf{v}$ be the velocity field of a flow in a domain D. For each point $\mathbf{x}_0$ in D, and unit vector $\mathbf{n}$ define the **curl of the flow about n at** $\mathbf{x}_0$

to be

$$\operatorname{curl} \mathbf{v}(\mathbf{x}_0, \mathbf{n}) = \lim_{r\to 0} \frac{\operatorname{circ}(C_r)}{r^2} \tag{8.19}$$

where C_r is the circle of radius r centered at $\mathbf{x}_0$ in the plane orthogonal to $\mathbf{n}$.

Example

7. Consider the flow (Figure 8.3)

$$x = x_0 \cos t + y_0 \sin t \qquad y = y_0 \cos t - x_0 \sin t \qquad z = z_0 + t$$

Let us take $\mathbf{x}_0 = (1, 0, 0)$ and $\mathbf{n} = \mathbf{E}_3$. Then, as we have already seen

$$\mathbf{v}(\mathbf{x}, t) = (y, -x, 1)$$

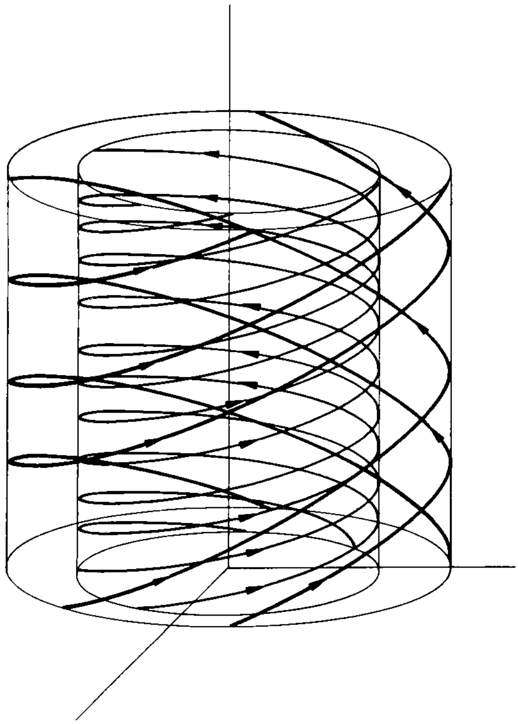

Figure 8.3

If we take

$$C_r = \left\{x - 1 + r\cos\frac{s}{r},\ y = r\sin\frac{s}{r},\ z = 0\right\}$$

then

$$\begin{aligned}
\operatorname{circ}(C_r) &= \int_{C_r} \langle \mathbf{v}(\mathbf{x}) - \mathbf{v}(1, 0, 0), \mathbf{T}\rangle\, ds \\
&= \int_0^{2\pi} \left\langle \left(r\sin\frac{s}{r} - r\cos\frac{s}{r}, 0\right), (-\sin s, \cos s, 0)\right\rangle ds \\
&= \int_0^{2\pi} (-r^2) r\, d\theta = -2\pi r^2
\end{aligned}$$

Thus the xy plane rotates around $(1, 0, 0)$ in the negative sense (with constant angular velocity), as t changes. If now we take $\mathbf{n} = \mathbf{E}_1$, we have

$$C_r = \left\{x = 1,\ y = r\cos\frac{s}{r},\ z = r\sin\frac{s}{r}\right\}$$

$$\begin{aligned}
\operatorname{circ}(C_r) &= \int_0^{2\pi} \left\langle \left(r\cos\frac{s}{r}, -1, 1\right), \left(0, -r\sin\frac{s}{r}, r\cos\frac{s}{r}\right)\right\rangle ds \\
&= 0
\end{aligned}$$

Thus there is no rotation in this plane.

Now, we shall compute the curl explicitly in terms of the velocity field $\mathbf{v}$. Again take $\mathbf{x}_0 = \mathbf{0}$ and let $\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$, $\boldsymbol{\beta} = (\beta^1, \beta^2, \beta^3)$ be two unit vectors in the plane orthogonal to $\mathbf{n}$ so that $\boldsymbol{\alpha} \to \boldsymbol{\beta} \to \mathbf{n}$ is a right-handed orthonormal basis. Thus $\mathbf{n} = \boldsymbol{\alpha} \times \boldsymbol{\beta}$, so

$$\mathbf{n} = (\alpha^2\beta^3 - \alpha^3\beta^2, \alpha^3\beta^1 - \alpha^1\beta^3, \alpha^1\beta^2 - \alpha^2\beta^1) \tag{8.20}$$

For a time we shall compute relative to this basis. C_r has this parametrization

$$\mathbf{x} = \mathbf{x}(s) = r\cos\frac{s}{r}\cdot\boldsymbol{\alpha} + r\sin\frac{s}{r}\cdot\boldsymbol{\beta} \tag{8.21}$$

The tangent vector is

$$\mathbf{T}(s) = -\sin\frac{s}{r}\cdot\boldsymbol{\alpha} + \cos\frac{s}{r}\cdot\boldsymbol{\beta}$$

Expanding the velocity field in terms of this basis:

$$\mathbf{v}(\mathbf{x}, 0) - \mathbf{v}(\mathbf{0}, 0) = v^{\alpha}(\mathbf{x})\boldsymbol{\alpha} + v^{\beta}(\mathbf{x})\boldsymbol{\beta} + v^{n}(\mathbf{x})\mathbf{n}$$

Then

$$\begin{aligned}\operatorname{circ}(C_r) &= \int_{C_r} \langle \mathbf{v}(\mathbf{x}, 0) - \mathbf{v}(\mathbf{0}, 0), \mathbf{T}(\mathbf{x})\rangle\, ds \\ &= \int_0^{2\pi} \left(-v^{\alpha}(\mathbf{x}(s))\sin\frac{s}{r} + v^{\beta}(\mathbf{x}(s))\cos\frac{s}{r}\right) ds \qquad (8.22)\end{aligned}$$

Now, substitute $\theta = s/r$ in the integral and approximate the v^{ν} ($\nu = \alpha, \beta$) by their differentials:

$$v^{\nu}(\mathbf{x}(\theta)) = v^{\nu}(\mathbf{0}) + dv^{\nu}(\mathbf{0})(\mathbf{x}(\theta)) + \varepsilon^{\nu}(\|\mathbf{x}\|)$$

where

$$\|\mathbf{x}\|^{-1}\varepsilon^{\nu}(\mathbf{x}) \to 0 \qquad \text{as } \|\mathbf{x}\| \to 0 \qquad (8.23)$$

Since $v^{\nu}(\mathbf{0}) = 0$, using (8.21) for $\mathbf{x}(\theta)$, we have

$$v^{\nu}(\mathbf{x}(\theta)) = r\cos\theta\cdot dv^{\nu}(\mathbf{0})(\boldsymbol{\alpha}) + r\sin\theta\cdot dv^{\nu}(\mathbf{0})(\boldsymbol{\beta}) = \varepsilon^{\nu}(\|\mathbf{x}\|)$$

Substituting these expressions into (8.22), we obtain

$$\begin{aligned}\operatorname{circ}(C_r) = &\int_0^{2\pi} [-dv^{\alpha}(\mathbf{0})(\boldsymbol{\alpha}) + dv^{\beta}(\mathbf{0})(\boldsymbol{\beta})]r^2\cos\theta\sin\theta\, d\theta \\ &+ \int_0^{2\pi} [-dv^{\alpha}(\mathbf{0})(\boldsymbol{\beta})\sin^2\theta + dv^{\beta}(\mathbf{0})(\boldsymbol{\alpha})\cos^2\theta]r^2\, d\theta \\ &+ \int_0^{2\pi} (-\varepsilon^{\alpha}(\mathbf{x})\cos\theta + \varepsilon^{\beta}(\mathbf{x})\sin\theta)r\, d\theta \\ = &\ \pi r^2[-dv^{\alpha}(\mathbf{0})(\boldsymbol{\beta}) + dv^{\beta}(\mathbf{0})(\boldsymbol{\alpha})] \\ &+ r\int_0^{2\pi} [-\varepsilon^{\alpha}(\mathbf{x})\cos\theta + \varepsilon^{\beta}(\mathbf{x})\sin\theta]\, d\theta\end{aligned}$$

Dividing by πr^2, and letting $r \to 0$, the second term disappears because of (8.23) and we obtain

$$\operatorname{curl} \mathbf{v}(\mathbf{0}, \mathbf{n}) = dv^{\beta}(\mathbf{0})(\boldsymbol{\alpha}) - dv^{\alpha}(\mathbf{0})(\boldsymbol{\beta}) \tag{8.24}$$

This can be rewritten in terms of the vector $\mathbf{n}$. Let $\mathbf{v} = (v^1, v^2, v^3)$ in terms of the standard Euclidean coordinates. Then

$$v^{\alpha}(\mathbf{x}, 0) = \langle \mathbf{v}(\mathbf{x}, 0) - \mathbf{v}(\mathbf{0}, 0), \boldsymbol{\alpha} \rangle = \sum_{i=1}^{3} [v^i(\mathbf{x}, 0) - v^i(\mathbf{0}, 0)]\alpha^i$$

so

$$dv^{\alpha}(\mathbf{0})(\boldsymbol{\beta}) = \sum_{i=1}^{3} \frac{\partial v^i}{\partial x^j} \beta^j \alpha^i$$

Similarly,

$$dv^{\beta}(\mathbf{0})(\boldsymbol{\alpha}) = \sum_{i=1}^{3} \frac{\partial v^i}{\partial x^j} \alpha^j \beta^i$$

(8.24) can be expanded out as

$$\begin{aligned}
\operatorname{curl} \mathbf{v}(\mathbf{0}, \mathbf{n}) &= \sum_{i=1}^{3} \frac{\partial v^i}{\partial x^j} \alpha^j \beta^i - \sum_{i=1}^{3} \frac{\partial v^i}{\partial x^j} \beta^j \alpha^i \\
&= \sum_{i=1}^{3} \frac{\partial v^i}{\partial x^j} (\alpha^j \beta^i - \alpha^i \beta^j) \\
&= \left(\frac{\partial v^2}{\partial x^3} - \frac{\partial v^3}{\partial x^2}\right)(\alpha^2 \beta^3 - \alpha^3 \beta^1) + \left(\frac{\partial v^3}{\partial x^1} - \frac{\partial v^1}{\partial x^3}\right)(\alpha^3 \beta^1 - \alpha^1 \beta^3) \\
&\quad + \left(\frac{\partial v^1}{\partial x^2} - \frac{\partial v^2}{\partial x^1}\right)(\alpha^1 \beta^2 - \alpha^2 \beta^1)
\end{aligned} \tag{8.25}$$

Referring back to (8.20) we see that this is the inner product of a vector derived from $\mathbf{v}$ with the given unit vector $\mathbf{n}$. We collect these results in a definition and a proposition.

Definition 3. If $\mathbf{v} = (v^1, v^2, v^3)$ is a vector field defined in a domain in R^3, we defined the vector field curl $\mathbf{v}$ by

$$\operatorname{curl} \mathbf{v} = \left(\frac{\partial v^2}{\partial x^3} - \frac{\partial v^3}{\partial x^2}, \frac{\partial v^3}{\partial x^1} - \frac{\partial v^1}{\partial x^3}, \frac{\partial v^1}{\partial x^2} - \frac{\partial v^2}{\partial x^1}\right) \tag{8.26}$$

Proposition 2. *If* $\mathbf{v}$ *is the velocity field of a fluid flow, the curl of* $\mathbf{v}$ *at* $\mathbf{x}_0$ *around the direction* $\mathbf{n}$ *at time t is given by* curl $\langle \mathbf{v}(\mathbf{x}_0, t), \mathbf{n} \rangle$.

Proof. Equation (8.25) is just $\langle \text{curl } \mathbf{v}, \mathbf{n} \rangle$.

Definition 4. A flow with velocity field $\mathbf{v}$ is called **irrotational** if curl $\mathbf{v} = \mathbf{0}$.

Examples

8. Let $\mathbf{v}(\mathbf{x}) = (-y, x, 1)$ (as in Example 6). Then

$$\text{curl } \mathbf{v} = (0, 0, -2)$$

Thus for any plane $\Pi = \{\mathbf{p}: \langle \mathbf{p} - \mathbf{x}, \mathbf{n} \rangle = 0\}$ through $\mathbf{x}$, the rotation in that plane has angular velocity $-2\langle \mathbf{n}, \mathbf{E}_3 \rangle$. Thus the maximum rotation is about the z axis.

In general, curl $\mathbf{v}(\mathbf{x})$ spans the axis of the "infinitesimal" rotation about $\mathbf{x}$ and its magnitude is the angular velocity.

9. Let

$$x = x_0(1 + t) + y_0(1 - e^t) \qquad y = y_0 e^{-t} \qquad z = z_0(1 + t)$$

be the equations of a flow. The velocity field is

$$\mathbf{v}(\mathbf{x}, t) = \left(\frac{x - ye^t - (2 + t)ye^{2t}}{1 + t}, -y, \frac{z}{1 + t} \right)$$

thus

$$\text{curl } \mathbf{v}(\mathbf{x}, t) = \left(0, 0, \frac{-e^t - (2 + t)e^{2t}}{1 + t} \right)$$

so again the rotation at any point is about the z axis. Notice that the equations break down at $t = -1$. We can consider that as the initial point of the motion: the fluid came, at $t = -1$ spinning off the xy plane with infinite angular velocity.

The form of curl $\mathbf{v}$ recalls the discussion of closed and exact forms in the previous chapter. If we consider the differential 1-form $\omega = \langle \mathbf{v}, d\mathbf{x} \rangle$ associated to the vector field $\mathbf{v}$, then curl $\mathbf{v} = \mathbf{0}$ is the necessary condition for

ω to be the differential of a function (and by Poincaré's lemma it is locally sufficient). In particular, *if the field is conservative, then the flow induced by the field is irrotational.*

We can make physical sense of this statement by referring it to the acceleration field $\mathbf{a} = \partial \mathbf{v}/\partial t$ of the flow rather than the velocity field. By Newton's law this is essentially the field of forces which generates the flow. As we have seen, if this field is conservative, then the work done by the flow in moving a mass from one point to another is precisely what is needed; it is the same as the change in energy level. For this to be the case no work can be expended in wastelessly rotating the mass; hence the field is irrotational.

In the theory of electromagnetism the existence of two fields, the electric $\mathbf{E}$, and the magnetic $\mathbf{H}$, is postulated. Certain relations between these fields, corroborated by experimental evidence form the basic laws of the subject. These are Maxwell's equations. Two of these are

$$\operatorname{curl} \mathbf{E} + \sigma \frac{\partial \mathbf{H}}{\partial t} = \mathbf{0}, \qquad \operatorname{div} \mathbf{H} = 0$$

(σ a suitable constant), which state that the rate of change of the magnetic field is determined by the rotation of the electric field, and that the "magnetic flow" is incompressible.

Here are several important relations between the gradient, curl, and divergence which are easily derived.

$$\operatorname{curl} \nabla f = \mathbf{0} \tag{8.27}$$

$$\operatorname{div} \operatorname{curl} \mathbf{v} = 0 \tag{8.28}$$

$$\operatorname{div} \nabla f = \Delta f \tag{8.29}$$

$$\operatorname{curl} f\mathbf{v} = f \operatorname{curl} \mathbf{v} + \nabla \mathbf{f} \times \mathbf{v} \tag{8.30}$$

$$\operatorname{div} (f\mathbf{v}) = f \operatorname{div} \mathbf{v} + \langle \nabla f, \mathbf{v} \rangle \tag{8.31}$$

Example

10. Suppose

$$\mathbf{A} = (-x, 0, y)$$

is the acceleration field of a fluid in motion. Find the equations of motion, assuming an initial velocity field of $(0, 1, 0)$, and find the divergence and curl of the flow.

If $\mathbf{x} = \boldsymbol{\phi}(\mathbf{x}_0, t)$ is the equation of motion, we have

$$\boldsymbol{\phi}(\mathbf{x}_0, 0) = \mathbf{x}_0$$

$$\frac{\partial \phi}{\partial t}(\mathbf{x}_0, 0) = (0, 1, 0)$$

and $\boldsymbol{\phi}(\mathbf{x}_0, t)$ solves the differential equation

$$(x, y, z)'' = (-x, 0, y)$$

The general solutions are

$$x = A_0 \cos t + B_0 \sin t$$

$$y = A_1 + B_1 t$$

$$z = A_2 + B_2 t + \frac{A_1}{2} t^2 + \frac{B_1}{3} t^3$$

The initial conditions give these as the equations of motion:

$$x = x_0 \cos t$$

$$y = y_0 + t$$

$$z = z_0 + y_0 \frac{t^2}{2} + \frac{t^3}{3}$$

The velocity field is

$$\mathbf{V}(\mathbf{x}, t) = (-x \tan t, 1, ty)$$

$$\operatorname{div} \mathbf{V}(\mathbf{x}, t) = -\tan t$$

$$\operatorname{curl} \mathbf{V}(\mathbf{x}, t) = (-t, 0, 0)$$

Notice that at $t = \pi/2$ the holocaust arrives. Before that moment, our fluid is moving generally in the positive y direction, rotating clockwise around the line parallel to the x axis and spinning away from it ($t < 0$) and back again toward it when $t > 0$.

• EXERCISES

9. Compute the curl for these fluid flows:
   (a) $x = x_0 + ty_0 \qquad y = y_0 e^t - tz_0 \qquad z = z_0 e^{-t} + tx_0$
   (b) $\mathbf{v}(x, y, z) = (-z, x, y)$
   (c) $\mathbf{v}(x, y, z) = (y, z, x)$
   (d) The flow described in Exercise 6(b).
   (e) The flow of Exercise 6(c).
   (f) The flow of Exercise 6(e).
10. Verify Equations (8.27)–(8.31).
11. Find the equations of motion and analyze the flow as in Example 8 given this acceleration field and initial velocity:
   (a) $\mathbf{A} = (-y, x, 1) \qquad \mathbf{V}(\mathbf{x}_0) = 0$
   (b) $\mathbf{A} = (x, z, x) \qquad \mathbf{V}(\mathbf{x}_0) = (0, 0, 1)$
12. Compute the rotation at $\mathbf{x}_0$ about the $\mathbf{E}_2$ axis for the flow of Example 6.

• PROBLEMS

7. Suppose we are given a time-independent field of forces $\mathbf{F}$ in a medium of constant density (say $= 1$). By Newton's law the fluid will flow according to the equation $\mathbf{F} = \mathbf{A}$. Let D be a small ball of fluid. The kinetic energy of D at time t is

$$\frac{1}{2}\int_{D_t} \|\mathbf{v}\|^2 \, dV$$

where $\mathbf{v}$ is the velocity field of the flow. Show that the work done by $\mathbf{F}$ in moving D to D_k is equal to the change in kinetic energy. (*Hint*:

$$\partial/\partial t(\|\mathbf{v}\|^2) = \langle \mathbf{v}, \mathbf{F} \rangle.)$$

8. Verify these identities:
   (a) $\operatorname{curl} g\nabla f = \nabla g \times \nabla f$
   (b) $\operatorname{curl} f\nabla f = 0$
9. Show that if $\mathbf{u}$, $\mathbf{v}$ are curl-free vector fields, then $\mathbf{u} \times \mathbf{v}$ is divergence free.
10. Show that in a ball, a vector field is a gradient if and only if its curl is zero.
11. Let $\mathbf{M}$ be a 3×3 matrix, and consider the flow

$$\mathbf{x} = \exp(\mathbf{M}t)\mathbf{x}_0$$

   (a) Compute the divergence and curl of the velocity field of the flow.
   (b) Show that the flow is divergence free if and only if $\operatorname{tr} \mathbf{M} = 0$
   (c) Show that the flow is curl free if and only if $\mathbf{M}$ is symmetric.

12. Consider the flow

$$\mathbf{x} = \exp(\mathbf{M}t)\mathbf{x}_0$$

where $\mathbf{M}$ is a symmetric matrix

(a) Show that the velocity field of the flow is conservative and has the potential function

$$\Pi(\mathbf{x}) = \frac{1}{2}\langle \mathbf{M}\mathbf{x}, \mathbf{x}\rangle$$

(b) Show that the flow in an eigenspace with eigenvalue a is in a straight line either toward the origin ($a < 0$), or away from the origin ($a > 0$).

(c) Diagram the flow lines for such a flow in the plane in case the eigenvalues (i) are the same; (ii) have the same sign; (iii) have opposite signs.

8.3 Surfaces

A surface in R^3 is (as we have been using the notion in this text) a subset of R^3 which is two dimensional. By this we mean that every point has some neighborhood which can be put into one-to-one correspondence with a domain in the plane. We shall assume that this correspondence is smooth. It is given by a continuously differentiable mapping with a nonsingularity condition on its differential.

Definition 5. A **surface patch** in R^3 is the image of a domain D in R^2 under a map $\mathbf{x} = \mathbf{x}(u, v)$ with these properties:

(i) $\mathbf{x}$ is one-to-one.
(ii) $\mathbf{x}$ is continuously differentiable.
(iii) The vectors $\partial\mathbf{x}/\partial u$, $\partial\mathbf{x}/\partial v$ are independent at every point. (u, v) are called the parameters for the surface patch. The curves $u =$ constant, and $v =$ constant are called the *parametric curves.*

A surface is a set Σ in R^3 which can be covered by surface patches, that is, every point $\mathbf{p}$ on Σ has a neighborhood N such that $\Sigma \cap N$ is a surface patch.

Notice that if we fix $u = c$, then the function $\boldsymbol{\phi}(v) = \mathbf{x}(c, v)$ parametrizes a curve (since ϕ is also one-to-one and

$$\frac{d\phi}{dv} = \frac{\partial \mathbf{x}}{\partial v}$$

is everywhere nonzero). The vector $\partial \mathbf{x}/\partial v$ is thus the tangent vector to the parametric curve u = constant. Condition (iii) asks that the curves $u = c$, $v = c'$ at any point have independent tangents. Another way of phrasing (iii) is that the 2×3 matrix

$$\begin{pmatrix} \dfrac{\partial \mathbf{x}}{\partial u} \\ \\ \dfrac{\partial \mathbf{x}}{\partial v} \end{pmatrix}$$

has rank 2.

Examples

11. The sphere: $x^2 + y^2 + z^2 = 1$ (Figure 8.4). Near the point $(0, 0, 1)$ we can write z as a function of x and y on the plane: $z = (1 - x^2 - y^2)^{1/2}$. Thus we can use x, y to define a surface patch surrounding $(0, 0, 1)$:

$$\mathbf{x} = \mathbf{x}(u, v) = (u, v, (1 - u^2 - v^2)^{1/2})$$

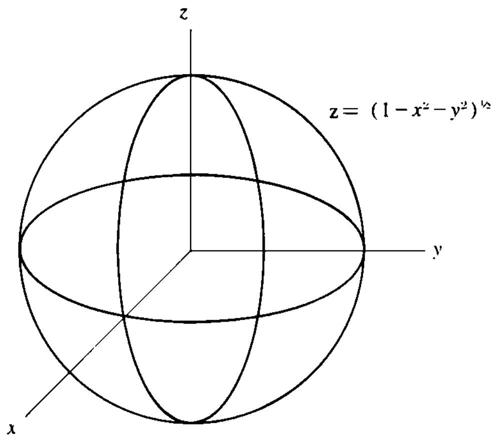

Figure 8.4

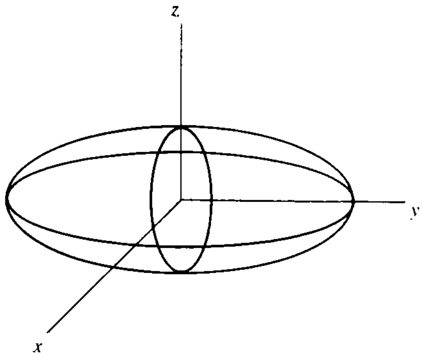

Figure 8.5

which coordinatizes the upper hemisphere as u, v range through the disk $u^2 + v^2 < 1$. Since

$$\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u} = (1, 0, -u(1 - u^2 - v^2)^{-1/2})$$

$$\mathbf{x}_v = \frac{\partial x}{\partial v} = (0, 1, -v(1 - u^2 - v^2)^{-1/2})$$

these vectors are independent. Every point on the sphere can be put in such a surface patch, by permuting the roles of (x, y, z) above. For example, the point $(-1, 0, 0)$ lies in the surface patch given by

$$\mathbf{x} = \mathbf{x}(u, v) = (-(1 - u^2 - v^2)^{1/2}, u, v) \qquad u^2 + v^2 < 1$$

Spherical coordinates can be used to coordinatize the whole sphere except for the points $(0, 0, \pm 1)$:

$$\mathbf{x} = \mathbf{x}(\theta, \phi) = (\cos\theta \cos\phi, \cos\theta \sin\phi, \sin\theta)$$

12. The ellipsoid (Figure 8.5)

$$a^2x^2 + b^2y^2 + c^2z^2 = 1$$

is also easily parametrized by spherical coordinates (again except for $z = \pm c^{-1}$):

$$\mathbf{x} = \mathbf{x}(u, v) = \left(\frac{\cos u \cos v}{a}, \frac{\cos u \sin v}{b}, \frac{\sin u}{c}\right)$$

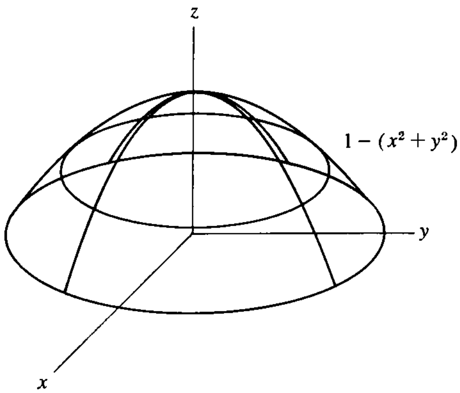

Figure 8.6

13. The paraboloid $z = x^2 + y^2$ (Figure 8.6) is a surface patch: it is coordinated by

$$\mathbf{x} = \mathbf{x}(u, v) = (u, v, u^2 + v^2)$$

Since $\mathbf{x}_u = (1, 0, 2u)$, $\mathbf{x}_v = (0, 1, 2v)$, they are independent.

14. The cone $z = (x^2 + y^2)^{1/2}$ (Figure 8.7) can be coordinatized, except for the vertex, by

$$\mathbf{x} = \mathbf{x}(u, v) = (u, v, (u^2 + v^2)^{1/2} \qquad u \neq 0, \quad v \neq 0$$

We might ask if there is any way to coordinatize a neighborhood of the vertex of the cone. It is quite difficult to show that there exists no function which does so, but there is one important implication of the differentiability of such a function which is easy to check out. The differentiability implies good approximability by linear functions, thus we should anticipate the existence of a linear surface (a plane) which comes "nearest" the surface at a given point. This is the *tangent plane*; which we shall now describe by limiting arguments as in the case of the tangent line to a curve.

Suppose $\mathbf{p}$ is a point on a surface Σ and $\mathbf{q}$, $\mathbf{r}$ are two nearby points. The three points $\mathbf{p}, \mathbf{q}, \mathbf{r}$ (in general) determine a plane. As $\mathbf{q}, \mathbf{r}$ tend to $\mathbf{p}$, this plane will (in general) attain a limiting position: this is the tangent plane. We now compute this process with coordinates. Suppose the function $\mathbf{x} = \mathbf{x}(u^1, u^2)$, $(u^1, u^2) \in D$ coordinatizes Σ near $\mathbf{p}$. We may assume $\mathbf{p} = \mathbf{x}(0, 0) = \mathbf{0}$. Let $\mathbf{q} = \mathbf{x}(u^1, u^2)$, $\mathbf{r} = x(v^1, v^2)$. The plane $\Pi(\mathbf{q}, \mathbf{r})$ through $\mathbf{p}, \mathbf{q}, \mathbf{r}$ is then the set of all vectors perpendicular to

$$\mathbf{q} \times \mathbf{r} = \mathbf{x}(u^1, u^2) \times \mathbf{x}(v^1, v^2) \tag{8.32}$$

In order to take the limit we approximate $\mathbf{x}$ by its differential

$$\mathbf{x}(u^1, u^2) = \mathbf{x}_1(0)u^1 + \mathbf{x}_2(0)u^2 + \varepsilon(\|\mathbf{u}\|)$$

where $t^{-1}\varepsilon(t) \to 0$ as $t \to 0$. Equation (8.32) becomes

$$\mathbf{q} \times \mathbf{r} = (\mathbf{x}_1(0) \times \mathbf{x}_2(0))(u^1v^2 - u^2v^1) + \mathbf{R} \tag{8.33}$$

where we have combined all the error terms in the expression $\mathbf{R}$. The important behavior of $\mathbf{R}$ is this:

$$\mathbf{R}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u}\|\varepsilon_1(\|\mathbf{v}\|) + \|\mathbf{v}\|\varepsilon_2(\|\mathbf{u}\|) + \varepsilon_3(\|\mathbf{u}\|)\varepsilon(\|\mathbf{v}\|)$$

where the ε_i all have the same behavior: $t^{-1}\varepsilon(t) \to 0$ as $t \to 0$.

Now, so as to treat the remainder $\mathbf{R}$ as an insignificant remainder, we must be careful with the term $u^1v^2 - u^2v^1$. It may, for example, be zero, in which case the remainder becomes very significant. Thus we must assume that

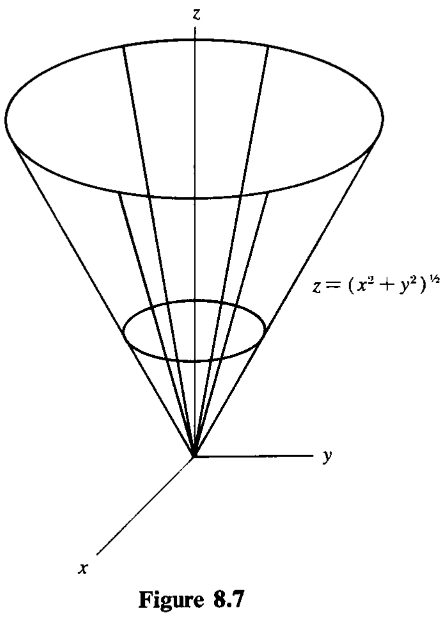

Figure 8.7

this terms tends to zero more slowly than $\mathbf{R}$ as $\mathbf{q}, \mathbf{r} \to \mathbf{p}$. Since

$$u^1v^2 - u^2v^1 = \sin \phi(\|\mathbf{u}\|)(\|\mathbf{v}\|)$$

it suffices to assume that the *angle between the coordinate vectors* does not tend to zero as $\mathbf{q}, \mathbf{r} \to \mathbf{p}$. Then, under this assumption, we can divide (8.33) by $u^1v^2 - u^2v^1$, obtaining $\Pi(\mathbf{q}, \mathbf{r})$ as the plane through $\mathbf{p}$ orthogonal to the vector

$$\mathbf{x}_1(0) \times \mathbf{x}_2(0) + \mathbf{R}^1$$

where $\mathbf{R}^1 \to 0$ as $\mathbf{q}, \mathbf{r} \to \mathbf{p}$. Thus the limiting position of $\Pi(\mathbf{q}, \mathbf{r})$ is the plane orthogonal to $(\partial\mathbf{x}/\partial u^1) \times (\partial\mathbf{x}/\partial u^2)$ at $\mathbf{p}$: it is the plane spanned by

$$\frac{\partial\mathbf{x}}{\partial u^1}(0), \frac{\partial\mathbf{x}}{\partial u^2}(0)$$

Definition 6. Let $\mathbf{p}$ be a point on a surface Σ coordinatized by $\mathbf{x} = \mathbf{x}(u^1, u^2)$. The **tangent plane to Σ at $\mathbf{p}$** is the plane spanned by the vectors $\partial\mathbf{x}/\partial u^1$, $\partial\mathbf{x}/\partial u^2$ at $\mathbf{p}$.

Proposition 3. *Let $\mathbf{p}$ be a point on the surface Σ, and let $\Pi(\mathbf{q}, \mathbf{r})$ be the plane spanned by two points $\mathbf{q}, \mathbf{r}$ on Σ so that the angle between $\mathbf{q} - \mathbf{p}$ and $\mathbf{r} - \mathbf{p}$ is nonzero. If $\mathbf{q}, \mathbf{r} \to \mathbf{p}$ so that this angle remains bounded away from zero, then $\Pi(\mathbf{q}, \mathbf{r})$ tends to the plane tangent to Σ at $\mathbf{p}$.*

Of course the angle assumption is crucial, Problem 28 exhibits the difficulty obtained without it.

Examples

15. There is no tangent plane to the cone

$$z = (x^2 + y^2)^{1/2}$$

at its vertex (Figure 8.7). For, if we take $\mathbf{q}_1 = (t, 0, t)$, $\mathbf{q}_2 = (0, t, t)$, the plane spanned by $\mathbf{q}_1$ and $\mathbf{q}_2$ is the plane spanned by $(1, 0, 1)$, $(0, 1, 1)$ for all $t \to 0$. Thus this is a candidate for the tangent plane. However, if we consider now the points $\tilde{\mathbf{q}}_1 = (-t, 0, t)$, $\tilde{\mathbf{q}}_2 = (0, -t, t)$ for $t \geq 0$, the candidate we obtain is the plane spanned by $(-1, 0, 1)$, $(0, -1, 1)$. Since these two planes are distinct, there can be no tangent plane (Figure 8.8).

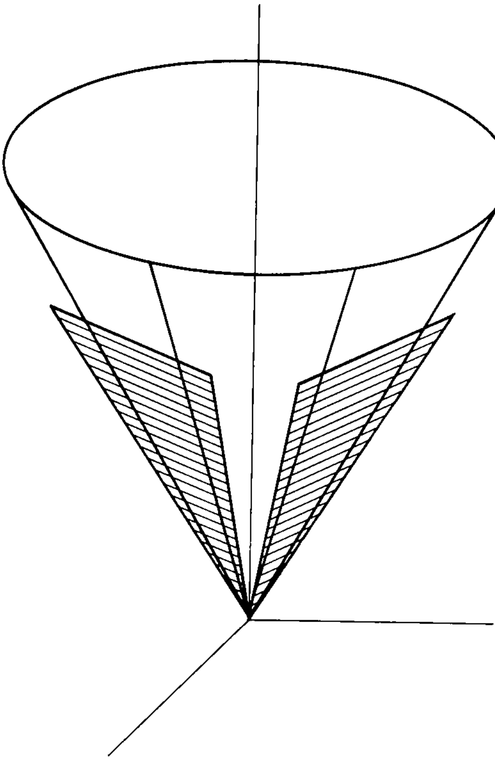

Figure 8.8

16. The cylinder $x^2 + y^2 = 1$ is a surface. It can be coordinatized by using cylindrical coordinates:

$$\mathbf{x} = \mathbf{x}(u, v) = (\cos u, \sin u, v)$$
$$\mathbf{x}_u = (-\sin u, \cos u, 0)$$
$$\mathbf{x}_v = (0, 0, 1)$$

The tangent plane at $\mathbf{x}(u, v)$ is the plane orthogonal to the vector $\mathbf{x}_u \times \mathbf{x}_v = (\cos u, \sin u, 0)$.

17. If $\mathbf{x} = \mathbf{x}(s)$ is the equation of a curve, the "surface swept out" by its family of tangent lines is a surface. It is parametrized by

$$\mathbf{x} = \mathbf{x}(s, t) = \mathbf{x}(s) + t\mathbf{T}(s)$$

We have

$$\mathbf{x}_s = \mathbf{T}(s) + t\kappa\mathbf{N}(s) \qquad \mathbf{x}_t = \mathbf{T}(s)$$

Thus, so long as $\kappa \neq 0$, s, t are patch coordinates for all $s, t > 0$. This surface is called the **developable** defined by the curve. Its tangent plane at the point (s, t) is the same as the osculating plane to the curve at $\mathbf{x}(s)$.

Let Σ be a surface, and $\mathbf{p}$ a point on the surface. We shall denote the tangent plane to Σ at $\mathbf{p}$ by $\mathrm{T}(\mathbf{p})$. If $\mathbf{x} = \mathbf{x}(u, v)$ parametrizes Σ in a neighborhood of $\mathbf{p}$, with $\mathbf{p} = \mathbf{x}(u_0, v_0)$, then the vectors $\partial\mathbf{x}/\partial u(u_0, v_0)$, $\partial\mathbf{x}/\partial v(u_0, v_0)$ span the plane $T(\mathbf{p})$. The inner product on R^3 induces an inner product on this plane just by restriction. It will be valuable to us to see how to express this inner product in terms of the basis $\mathbf{x}_u, \mathbf{x}_v$. If $\mathbf{t} = a\mathbf{x}_u + b\mathbf{x}_v$ is a vector in $T(\mathbf{p})$ its length is given by

$$\|\mathbf{t}\|^2 = \langle \mathbf{t}, \mathbf{t} \rangle = a^2\langle \mathbf{x}_u, \mathbf{x}_v \rangle + 2ab\langle \mathbf{x}_u, \mathbf{x}_v \rangle + b^2\langle \mathbf{x}_u, \mathbf{x}_v \rangle$$

Suppose that C is a curve on Σ. Choose a parametrization of C:

$$\mathbf{x} = \mathbf{g}(s) \qquad 0 \leq s \leq L \tag{8.34}$$

Let $(u(s), v(s))$ be the (u, v) coordinates of $\mathbf{g}(s)$. Then (8.34) is the same as

$$\mathbf{x} = \mathbf{x}(u(s), v(s)) \tag{8.35}$$

and by the chain rule, the tangent to C is

$$\mathbf{T} = \mathbf{x}_u \frac{du}{ds} + \mathbf{x}_v \frac{dv}{ds}$$

and

$$\|\mathbf{T}\|^2 = \langle \mathbf{x}_u, \mathbf{x}_u \rangle \left(\frac{du}{ds}\right)^2 + 2\langle \mathbf{x}_u, \mathbf{x}_v \rangle \frac{du}{ds}\frac{dv}{ds} + \langle \mathbf{x}_v, \mathbf{x}_v \rangle \left(\frac{dv}{ds}\right)^2 \tag{8.36}$$

We shall use these following notational conventions relative to coordinates on Σ:

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle \qquad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle \qquad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle \tag{8.37}$$

In terms of this notation we have this way, intrinsic to the surface, for computing the lengths of curves on Σ:

Proposition 4. *Let Σ be a surface patch parametrized by $\mathbf{x} = \mathbf{x}(u, v)$. Let C be a curve on Σ parametrized by $\mathbf{x} = \mathbf{x}(u(t), v(t))$. $a \le t \le b$. Then the length of C is*

$$\int_a^b \left[E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\frac{dv}{dt}\right)^2 + G\left(\frac{dv}{dt}\right)^2\right]^{1/2} dt \tag{8.38}$$

Proof. The length of C is

$$\int_b^a \|\mathbf{T}\|\, dt$$

which is, by (8.36), given by (8.38).

We shall adopt the convention (borrowed from the differential form notation) that ds is the integrand which gives arc length along a curve. This means just that the length of any curve C is $\int_C ds$. According to (8.38) we can be assured that

$$ds = \left[E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2\right]^{1/2} dt$$

for any parameter t along C. We can also write this as

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 \tag{8.39}$$

Definition 7. The form (8.39), where E, F, G are given by (8.37) relative to a parametrization $\mathbf{x} = \mathbf{x}(u, v)$ on Σ is called the **first fundamental form of Σ.**

If C_1, C_2 are two curves given parametrically by

$$C_1\colon u = u_1(s) \qquad v = v_1(s)$$

$$C_2\colon u = u_2(s) \qquad v = v_2(s)$$

then their tangents are

$$\mathbf{T}_1 = \mathbf{x}_u \frac{du_1}{ds} + \mathbf{x}_v \frac{dv_1}{ds} \qquad \mathbf{T}_2 = \mathbf{x}_u \frac{du_2}{ds} + \mathbf{x}_v \frac{dv_2}{ds}$$

At a point of intersection **p** the vectors $\mathbf{T}_1(\mathbf{p})$, $\mathbf{T}_2(\mathbf{p})$ lie in the tangent plane at **p** and their inner product is

$$\langle \mathbf{T}_1, \mathbf{T}_2 \rangle = E\frac{du_1}{ds}\frac{du_2}{ds} + F\left(\frac{du_1}{ds}\frac{dv_2}{ds} + \frac{du_2}{ds}\frac{dv_1}{ds}\right) + G\frac{dv_1}{ds}\frac{dv_2}{ds}$$

The curves are **orthogonal at p** if $\langle \mathbf{T}_1, \mathbf{T}_2 \rangle = 0$

Proposition 5. *The parametric curves $u =$ constant, $v =$ constant on a surface patch are orthogonal if and only if $F = 0$.*

Proof. The tangent line to $u = c$ is spanned by $\mathbf{x}_v$; the tangent line to $v = c$ is spanned by $\mathbf{x}_u$. These lines are orthogonal if and only if $\langle \mathbf{x}_u, \mathbf{x}_v \rangle = F = 0$.

Examples

18. The plane $z = 0$. In the standard rectangular coordinates we have $ds^2 = dx^2 + dy^2$. If $x = f(t)$, $y = g(t)$, $0 \le s \le L$, is any curve joining a to b we have (as in Chapter 5) the length of L is

$$\int_0^L [f'(t)^2 + g'(t)^2]^{1/2}\, dt$$

If we parametrize this curve by x we obtain the length as

$$\int_{a_1}^{a_2} \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx$$

This is minimized when $dy/dx = 0$; that is, when the curve is a straight line. This conforms with known facts.

19. The cylinder

$$\mathbf{x} = \mathbf{x}(u, v) = (\cos u, \sin u, v)$$

Here $\mathbf{x}_u = (-\sin u, \cos u, 0)$, $\mathbf{x}_v = (0, 0, 1)$. Thus $E = 1 = G$, $F = 0$, so

$$ds^2 = du^2 + dv^2$$

Again, the length of a curve given as $v = v(u)$ is

$$\int_a^b \left[1 + \left(\frac{dv}{du}\right)^2\right]^{1/2} du$$

so the curves of minimal length (called *geodesics*) on the cylinder are those represented by straight lines in the u, v coordinates. Thus the typical geodesic on the cylinder is the helix

$$\mathbf{x} = (\cos t, \sin t, at)$$

20. For the sphere

$$\mathbf{x} = \mathbf{x}(u, v) = (\cos u \cos v, \cos u \sin v, \sin u)$$

we have $E = 1$, $F = 0$, $G = \cos^2 u$. Thus

$$ds^2 = du^2 + \cos^2 u \, dv^2$$

Once again, we discover the geodesics by minimizing the integral $\int_\gamma ds$. Let $\mathbf{a}, \mathbf{b}$ be two points on the sphere; by rotating the sphere we may suppose that $\mathbf{a}, \mathbf{b}$ lie on the longitude $v = 0$. If γ is any curve joining $\mathbf{a}$ to $\mathbf{b}$, the length of γ is

$$\int_{\mathbf{a}}^{\mathbf{b}} ds = \int_{\mathbf{a}}^{\mathbf{b}} (du^2 + \cos^2 u \, dv^2)^{1/2} \tag{8.40}$$

The length of the longitude ($u = 0$) is

$$\int_{\mathbf{a}}^{\mathbf{b}} (du^2)^{1/2} = \int_{\mathbf{a}}^{\mathbf{b}} du \tag{8.41}$$

Now (8.40) is always larger than (8.41) unless $dv = 0$ along γ; that is, v is constant. Thus it is the longitude which is the curve of the shortest distance between $\mathbf{a}$ and $\mathbf{b}$. By rotating back again we conclude that the geodesics on the sphere are the sections by diametric planes: the great circles.

Geodesics

The problem of finding the geodesics on any surface is more difficult, because the general form

$$E \, du^2 + 2F \, du \, dv + G \, dv^2$$

is harder to analyze. One way to proceed is to try to find coordinates so that the first fundamental form looks like the above examples: it has the form

$$ds^2 = du^2 + G\,dv^2 \tag{8.42}$$

When this is the case we can verify that the curves v = constant are geodesics (Problem 17). However, in order to find such coordinates, we must know what we are looking for; that is, we must know how to find geodesics in the first place. Thus, this line of reasoning has to be supplemented by the discovery of a characteristic property of geodesics. We seek such a characteristic property by trying to understand the "infinitesimal" behavior of a geodesic: this (we hope) leads to a differential equation which is solvable. Then we can carry out our original plan: solving the differential equations will provide a convenient coordinate system in which we can discover the curves of minimal length. We shall, however, not carry through the entire program here; we shall only derive the basic property.

If γ is a geodesic, a curve of minimal length, on the surface Σ, then, relative to Σ it is a straight line. That is, it would have to be as close to a straight line as it could be: it should bend only as much as it must in order to remain on Σ. Thus the rate of change of the tangent, *relative to* Σ, should be zero. Infinitesimally this says that the normal to the curve has no component on the tangent plane to Σ. We shall now show that a geodesic has this property.

Theorem 8.2. *Let γ be a geodesic (curve of minimal length) on the surface Σ. Then, at any point $\mathbf{p}$ on γ, the normal to γ is orthogonal to the tangent plane of Σ.*

Proof. Let $\mathbf{p} \in \gamma$ and let u, v be coordinates for Σ near $\mathbf{p}$ so that $\mathbf{p} = (u(0), v(0))$. We may choose these coordinates so that γ is the curve $v = 0$ and so that the coordinates are everywhere orthogonal (see Problems 9 and 10). Now let a be small enough so that the interval from $(-a, 0)$ to $(a, 0)$ in the uv plane lies on the domain D of the coordinates. If $\Gamma\colon v = f(u)$ defines a curve lying in D and joining $(-a, 0)$ to $(a, 0)$, then $\mathbf{x} = \mathbf{x}(u, f(u))$, $-a \le u \le a$ gives another curve on Σ, joining two points of γ (Figure 8.9). The length of Γ is no more than that of γ, since γ is a geodesic.

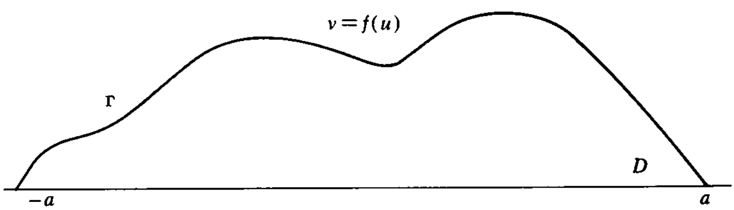

Figure 8.9

We have not yet done enough to investigate the local behavior of γ; we must consider a whole family of curves including γ rather than just one other. But that is easy to do: let Γ_t be the curve parametrized by

$$\Gamma_t\colon \mathbf{x} = \mathbf{x}(u, tf(u)) \qquad -a \le u \le a$$

for $-1 \le t \le 1$. γ is Γ_0 and Γ is Γ_1. Let $F(t)$ be the length of Γ_t. Then $F(t)$ has a minimum at $t = 0$, so (if it is differentiable) $F'(0) = 0$. We now compute this:

$$F(t) = \int_{-a}^{a} \|\mathbf{x}_u + \mathbf{x}_v tf'(u)\|\, du$$

is certainly a differentiable function of t, and

$$F'(t) = \int_{-a}^{a} \frac{\partial}{\partial t} \|\mathbf{x}_u + \mathbf{x}_v tf'(u)\|\, du$$

Now, at $t = 0$, the integrand is

$$\frac{\partial}{\partial t} \langle \mathbf{x}_u + \mathbf{x}_v tf'(u), \mathbf{x}_u + \mathbf{x}_v tf'(u)\rangle^{1/2}|_{t=0}$$

$$= \frac{1}{2}\frac{1}{\|\mathbf{x}_u\|} 2\langle \mathbf{x}_{uv} f(u) + \mathbf{x}_v f'(u), \mathbf{x}_u\rangle$$

$$= -\frac{\langle \mathbf{x}_{uu}, \mathbf{x}_v\rangle}{\|\mathbf{x}_u\|} f(u) \tag{8.43}$$

The last equation follows from the assumption that the coordinates are orthogonal: $\langle \mathbf{x}_v, \mathbf{x}_u\rangle = 0$. First, the second term drops out, secondly, the expression (8.43) derives from

$$0 = \frac{\partial}{\partial u} \langle \mathbf{x}_v, \mathbf{x}_u\rangle = \langle \mathbf{x}_{uv}, \mathbf{x}_u\rangle + \langle \mathbf{x}_v, \mathbf{x}_{uu}\rangle$$

Therefore, from $F'(0) = 0$, we obtain

$$\int_{-a}^{a} \frac{\langle \mathbf{x}_v, \mathbf{x}_{uu}\rangle}{\|\mathbf{x}_u\|} f(u)\, du = 0$$

This equation must hold for all differentiable functions f such that $f(-a) = f(a) = 0$. We conclude then that

$$\langle \mathbf{x}_v, \mathbf{x}_{uu}\rangle = 0$$

along γ (see Miscellaneous Problem 41 of Chapter 2). Now, the normal N to γ is in the plane spanned by $\mathbf{x}_u$ and $\mathbf{x}_{uu}$. Since these are both orthogonal to $\mathbf{x}_v$, $\mathrm{N} \perp \mathbf{x}_v$. Further, N is orthogonal to the tangent line of γ which is spanned by $\mathbf{x}_u$. Thus N is orthogonal to both $\mathbf{x}_u$ and $\mathbf{x}_v$, so is orthogonal to the tangent plane of Σ.

Examples

21. Find the geodesics on the surface

$$\Sigma: y = x^2$$

We parametrize Σ by $\mathbf{x} = \mathbf{x}(u, v) = (u, u^2, v)$. Let $u = u(s)$, $v = v(s)$ parametrize a geodesic Γ on Σ_1. Then Γ has the form

$$\mathbf{x} = (u(s), u^2(s), v(s))$$

and

$$\mathbf{x}_s = (u', 2uu', v')$$

$$\mathbf{x}_{ss} = \mathrm{N} = (u'', 2(u')^2 + 2uu'', v'')$$

For Γ to be a geodesic, this must be orthogonal to both

$$\mathbf{x}_u = (1, 2u, 0) \qquad \mathbf{x}_v = (0, 0, 1)$$

Thus, the functions $u(s)$, $v(s)$ parametrizing the geodesic Γ satisfy these differential equations

$$u'' + 2u[2(u')^2 + 2uu''] = 0$$

$$v'' = 0$$

Notice that from Picard's theorem the equations

$$u'' = \frac{-4u(u')^2}{1 + 4u^2}$$

$$v'' = 0$$

have unique solutions given the initial values of u, v, u', v'. Thus, there exists a curve of minimal length in every direction, at every point.

22. Find the geodesics on the cone

$$\Sigma: z^2 = x^2 + y^2$$

Notice that any plane $z + x \cos a + y \sin a = b$ intersects Σ at right angles (Figure 8.10). Thus the normal to the curve of intersection is orthogonal to the surface, and such a plane always intersects Σ in a geodesic. More generally, we can compute the equations for any geodesic using Theorem 8.2

$$\Sigma: \mathbf{x} = \mathbf{x}(u, v) = (v \cos u, v \sin u, v)$$

$$\mathbf{x}_u = (-v \sin u, v \cos u, 0)$$

$$\mathbf{x}_v = (\cos u, \sin u, 1)$$

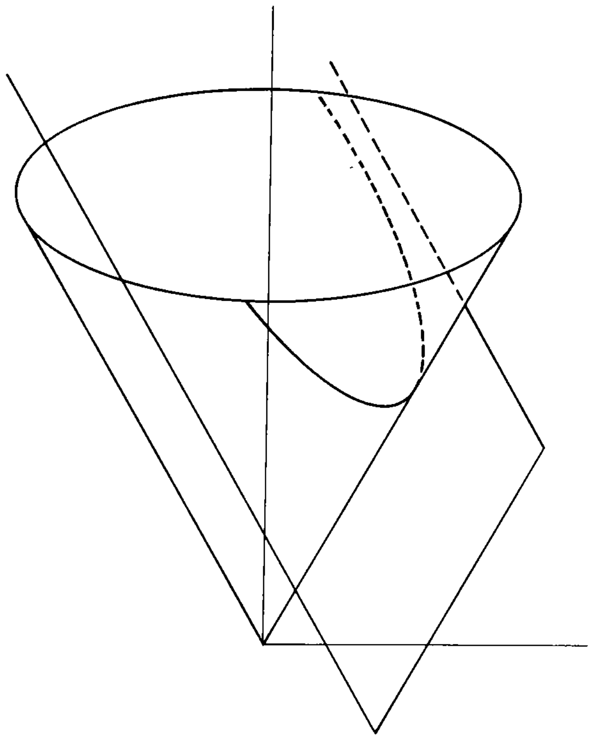

Figure 8.10

If $u = u(s)$, $v = v(s)$ parametrizes a geodesic Γ, then on Γ

$$\mathbf{x}_s = (v' \cos u - vu' \sin u, v' \sin u + vu' \cos u, v')$$

$$\begin{aligned}\mathbf{x}_{ss} = (&v'' \cos u - 2v'u' \sin u - vu'' \sin u - v(u')^2 \cos u,\\ &v'' \sin u + 2v'u' \cos u + vu'' \cos u - v(u')^2 \sin u, v'')\end{aligned}$$

The differential equations are readily computed (and hardly solved explicitly) by expressing $\langle \mathbf{x}_{ss}, \mathbf{x}_u \rangle = 0$, $\langle \mathbf{x}_{ss}, \mathbf{x}_v \rangle = 0$.

Surface Area

We would like now to define the area of a surface in a way analogous to the definition of the length of a curve. We select a collection of points $\mathbf{x}_1, \ldots, \mathbf{x}_k$ on Σ and replace Σ by the polygonal surface Σ' whose vertices are $\mathbf{x}_1, \ldots, \mathbf{x}_k$. If the points $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are very numerous and close to each other, then the sum of the areas of the faces of Σ' is a good approximation to the area of Σ. We can then try to define the area of Σ to be the limit of such sums as the set of points $\mathbf{x}_1, \ldots, \mathbf{x}_k$ becomes infinitely numerous and everywhere dense.

Now this definition unfortunately does not work, there are ways of so partitioning a surface so as to obtain any desired area (for a fuller account see Spivak, pp. 128–130). Rather than give it all up as a hopeless task because of this phenomenon, we try a different approach. First, we study the approximation of area in the small,hoping to generate a plausible formula for surface area (by *plausible* I mean that approximations to our formula are also approximations to our notion of area). If the formula turns out to be intrinsic, that is, independent of parametrizations, then it will define a relevant measure, which we shall call surface area. Returning to the above "approxi-

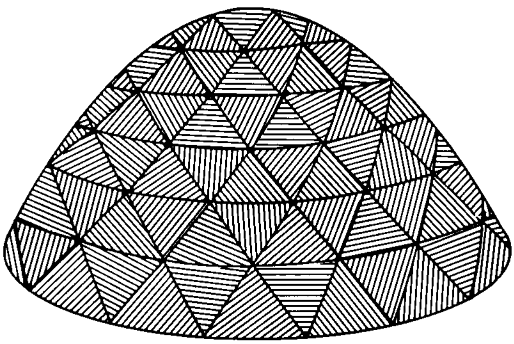

Figure 8.11

mation," let F be one of the faces of Σ', and $\mathbf{x}_0$ one of its vertices. Let F_0 be the projection of F onto Σ (see Figure 8.11) and F_t the projection onto the tangent plane $T(\mathbf{x}_0)$. If the surface is very smooth, then for small F these three surfaces have essentially the same area, and we can confuse the three. We may suppose that F_0 lies in a patch parametrized by $x = x(u, v)$ with $\mathbf{x}_0 = \mathbf{x}(u_0, v_0)$. Let D be such that

$$F = \{\mathbf{x}(u, v); (u, v) \in D\}$$

Confusing the surface with F_t we may take x to be the linear map

$$\mathbf{x}(u, v) = \mathbf{x}_0 + \mathbf{x}_u(u_0, v_0)u + \mathbf{x}_v(u_0, v_0)v$$

Now, we know how to compute area on the image of a linear map:

$$\text{area } (F_t) = \|\mathbf{x}_u(u_0, v_0) \times \mathbf{x}_v(u_0, v_0)\| \text{ area } D$$

This is true because it is true for rectangles, as we have seen in Proposition 28 of Chapter 1. Thus, at least on this coordinate patch, the area of Σ' is very close to

$$\Sigma \; \|\mathbf{x}_u(u_i, v_i) \times \mathbf{x}_v(u_i, v_i)\| \text{ area } (D_i)$$

where the $\{D_i\}$ partition the coordinate domain D and $(u_i, v_i) \in D_i$. The limit of such sums is

$$\int_D \|\mathbf{x}_u \times \mathbf{x}_v\| \, du \, dv$$

We take this to be the definition of surface area.

Definition 8. Let Σ be a surface patch with coordinates u, v, ranging through D in R^2. The **area of** Σ is

$$\int_D \|\mathbf{x}_u \times \mathbf{x}_v\| \, du \, dv$$

If Σ is a surface, partition Σ into pieces $D_1, \ldots, D_k$ such that each D_i is a surface patch. Define

$$\text{area } (\Sigma) = \sum_i \text{ area } (D_i)$$

We must show that this definition is independent of the particular partition.

Proposition 6. *The above definition is independent of the partition of Σ chosen.*

Proof. Suppose we also partition Σ another way: $\Sigma = E_1 \cup \cdots \cup E_n$. Then

$$\Sigma = (D_1 \cap E_1) \cup (D_1 \cap E_2) \cup \cdots \cup (D_k \cap E_n)$$

is still a third partition. Clearly,

$$\text{area}\,(E_i) = \sum \text{area}\,(D_j \cap E_i) \tag{8.44}$$

$$\text{area}\,(D_j) = \sum \text{area}\,(D_j \cap E_i) \tag{8.45}$$

since in each case we are computing relative to the same coordinates. We leave it to the reader (see Problem 18) to verify that the computation of the area of $D_j \cap E_i$ is the same whether it is done in the D_j or E_i coordinates. Then, summing (8.44) over i, and (8.45) over j, the right-hand sides are the same; and so are the lefts, as desired.

In accordance with our convention to denote ds as the integrand for arc length, we shall let dS denote the integrand for surface area. Thus, in terms of any coordinate system u, v we have $dS = H\,du\,dv$, where $H = \|\mathbf{x}_u \times \mathbf{x}_v\|$. It follows from Lagrange's identity (Chapter 1) that also $H = (EG - F^2)^{1/2}$.

Examples

23. Find the area of the sphere

$$\{x^2 + y^2 + z^2 = R^2\}$$

We use spherical coordinates:

$$\mathbf{x} = (R\cos u \cos v,\ R\cos u \sin v,\ R\sin u)$$

$$\mathbf{x}_u = (-R\sin u \cos v,\ -R\sin u \sin v,\ R\cos u)$$

$$\mathbf{x}_v = (-R\cos u \sin v,\ R\cos u \cos v,\ 0)$$

so $H = [EG - F^2]^{1/2} = R^2\,|\cos u|$. The area is

$$R^2 \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} \cos u\,du\,dv = 4\pi R^2$$

24. The area of the piece of the paraboloid is

$$\{z = x^2 + y^2, \qquad 0 \le z \le 1\}$$

The parametrization is

$$\mathbf{x} = (r \cos u, r \sin u, r)$$

$$\mathbf{x}_r = (\cos u, \sin u, 1) \qquad \mathbf{x}_u = (-r \sin u, r \cos u, 0)$$

$E = 2$, $F = 0$, $G = r^2$, $H = 2r$. The area is

$$\int_0^{2\pi} \int_0^1 2r\, dr\, d\theta = 2\pi$$

● EXERCISES

13. Let f be a C^1 function defined in a domain D in R^2.
    (a) Show that Σ: $\{z = f(x, y)\}$ is a surface patch with coordinate x, y.
    (b) Compute the first fundamental form and the area element for f.
    (c) Show that the element area is given by $\sec \gamma\, dx\, dy$, where γ is the angle between the normal to Σ and the z axis.
14. Find the tangent plane, first fundamental form and area element for these surfaces:
    (a) The paraboloid $x = y^2 + z^2$.
    (b) The cone $z^2 = x^2 + y^2$.
    (c) The hyperboloid $z = x^2 - y^2$.
    (d) Σ: $\mathbf{x}(u, v) = (u + v^2, v + u^2, uv)$
15. Find the length of the intersection of these surfaces:
    (a) $x^2 + y^2 + z^2 = 1$
    $\frac{1}{2}x^2 + 2y^2 + 2z^2 = 1$
    (b) $z^2 = 2x^2 + y^2$
    $z = x^2 + 2y^2$
16. Find the angle between the parametric curves at a general point for the surface given in Exercise 14(d).
17. Find the area cut off the tip of the paraboloid $x^2 = y^2 + z^2$ by the plane $x + z = 1$.
18. Find the area of these surfaces:
    (a) The cone $z^2 = x^2 + y^2 \qquad 0 \le z \le a$.
    (b) Σ: $x = (u, \cos u, v) \qquad 0 \le v \le \pi, \qquad -\pi \le u \le \pi$
    (c) The part of the hyperboloid $z = x^2 - y^2$ inside the unit ball.
    (d) The ellipsoid $x^2 + y^2 + 4z^2 = 4$.

• PROBLEMS

13. Recall that a differential form $M\,du + N\,dv$ determines a family of curves: those curves along which $M\,du + N\,dv = 0$. If $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$ is the first fundamental form of a surface patch show that the family of curves orthogonal to the family defined by $M\,du + N\,dv = 0$ is determined by

$$(EN - FM)\,du + (FN - GM)\,dv = 0$$

14. Let $\mathbf{p}_0$ be a point on the surface Σ. Show that we can find a surface patch near $\mathbf{p}_0$ so that the parametric curves are orthogonal. (*Hint:* Let u, v be coordinates near $\mathbf{p}_0$ and explicitly find the family of curves $u = u(t, c)$, $v = v(t, c)$ orthogonal to the curves $dv = 0$ such that $u(0, c) = 0$, $v(0, c) = v_0$. Show that v, c are orthogonal coordinates.)

15. Let γ be a curve on the surface Σ. Find orthogonal coordinates u, v at a point $\mathbf{p}_0$ on γ so that (i) γ is the curve $v = 0$, (ii) u is arc length along γ.

16. Show that a cube is not a surface along its edges.

17. Is ds a differential 1-form?

18. Find the differential equations for the geodesics on the torus (Figure 8.12):

$$x = (1 - \cos\phi)\sin\theta$$
$$y = (1 - \cos\phi)\cos\theta$$
$$z = \sin\phi$$

19. Find those planes which intersect the ellipse $x^2 + a^2y^2 + b^2z^2 = 1$ in a geodesic.

20. Let $\{(u, v) \in R^2\colon u > 0,\ v > 0\}$ parametrize a surface with first fundamental form $ds^2 = v^2\,du^2 + u^2\,dv^2$. Find the equation of the family of

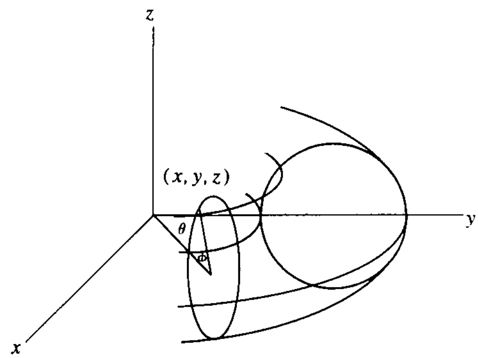

Figure 8.12

curves orthogonal to the curves $uv = \text{constant}$, and express the fundamental form in terms of these new coordinates.

21. Find the geodesics on the surface with first fundamental form $ds^2 = du^2 + f(u)\,dv^2$.

22. Show that the curves $v = \text{constant}$ on a surface with first fundamental form $E\,du^2 + G\,dv^2$ are geodesics if and only if $\partial E/\partial v = 0$.

23. Let Σ be a surface path with two different coordinates:

$$\Sigma: \mathbf{x} = \mathbf{x}(u, v) \qquad (u, v) \in D$$

$$\Sigma: \mathbf{x} = \mathbf{x}(r, s) \qquad (r, s) \in \Delta$$

Show that

$$\int_D \left\| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right\| du\,dv = \int_\Delta \left\| \frac{\partial \mathbf{x}}{\partial r} \times \frac{\partial \mathbf{x}}{\partial s} \right\| dr\,ds$$

(*Hint:* Define $u = u(s, t)$, $v = v(s, t)$ by this property: $\mathbf{x} = \mathbf{x}(r, s)$ if and only if $\mathbf{x} = \mathbf{x}(u, v)$ with $u = u(r, s)$, $v = v(r, s)$. Show that

$$\frac{\partial \mathbf{x}}{\partial r} \times \frac{\partial \mathbf{x}}{\partial s} = \left(\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right) \frac{\partial(u, v)}{\partial(r, s)}\Bigg)$$

The following problems use the **normal to a surface** Σ: this is a unit vector $\mathbf{N}$ orthogonal to the tangent plane.

24. Let γ be a curve on the surface Σ. Let $\mathbf{N}$ represent the normal to Σ, and $\mathbf{T}$ the tangent to γ. The *unit surface normal* to γ is the vector $\mathbf{N}_\gamma = \mathbf{N} \times \mathbf{T}$.

(a) Show that γ is a geodesic on Σ if and only if $\langle \mathbf{N}_\gamma, d\mathbf{T}/ds \rangle = 0$.

(b) In general, the inner produce $\kappa_g = \langle \mathbf{N}_\gamma, d\mathbf{T}/ds \rangle$ is called the geodesic curvature of γ on Σ. Suppose (u, v) are orthogonal coordinates on Σ and $\kappa_g{}^1$, $\kappa_g{}^2$ are the geodesic curvatures of the lines $v = \text{constant}$, $u = \text{constant}$, respectively. Verify **Liouville's formula**: the geodesic curvature of the curve γ is given by

$$\kappa_g = \frac{d\theta}{ds} + \kappa_g{}^1 \cos\theta + \kappa_g{}^2 \sin\theta$$

where θ is the angle between the tangent to γ and the direction $\mathbf{x}_u$. (*Hint:* Write

$$\mathbf{T} = \mathbf{T}_1 \cos\theta + \mathbf{T}_2 \sin\theta$$

where $\mathbf{T}_1$, $\mathbf{T}_2$ are the tangents to the curves $v = \text{constant}$, $u = \text{constant}$.

Then

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}_1}{ds}\cos\theta + \frac{d\mathbf{T}_2}{ds}\sin\theta + (-\mathbf{T}_1\sin\theta + \mathbf{T}_2\cos\theta)\frac{d\theta}{ds}$$

Substitute these expressions into

$$\kappa_g = \left(\frac{d\mathbf{T}}{ds}, \mathbf{N}\times\mathbf{T}\right)$$

and evaluate at $\theta = 0$, $\theta = \pi/2$.)

25. If γ is a curve on the surface Σ we can decompose $d\mathbf{T}/ds$ into its components tangent to and orthogonal to the surface:

$$\frac{d\mathbf{T}}{ds} = \kappa_g \mathbf{N}_\gamma + \kappa_N \mathbf{N}$$

where κ_N is called the **normal curvature** to $\mathbf{T}$.

(a) Show that the curvature of γ is $(\kappa_g{}^2 + \kappa_N{}^2)^{1/2}$.

(b) Show that the normal curvature of a curve γ depends only on the tangent to γ and is the same as the curvature of the curve of intersection of Σ with the plane through $\mathbf{T}$ and $\mathbf{N}$.

(c) Show that the curvature of the curve γ is given by $\kappa_N(\mathbf{T})\sec\theta$, where θ is the angle between $d\mathbf{T}/ds$ and $\mathbf{N}$.

26. Using Liouville's formula find the geodesic curvature of a general curve on the surface obtained by revolving the curve $z = \exp(-x^2)$ around the z axis.

27. Let Σ be a surface such that at every point every curve on Σ has zero normal curvature. Show that Σ is a piece of a plane.

28. Let $\mathbf{p}$ be a point on a surface Σ and let $\mathbf{q}$, $\mathbf{r}$ be two nearby points. It is possible to select $\mathbf{q}$, $\mathbf{r}$ tending to zero so that the plane determined by $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ does not converge to the tangent plane (unless Σ is itself a plane). For example, if γ is the curve intersection of some plane Π with Σ and if $\mathbf{r}$ follows $\mathbf{q}$ along γ then the plane determined by $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ is always Π, which need not be the tangent plane to Σ. Furthermore, if we move $\mathbf{q}$ slightly off γ we can be sure of the same behavior with the requirement that the angle between $\mathbf{q}$ and $\mathbf{r}$ (in some parametrization) is not zero (however, it must tend to zero). Here is an explicit example. Σ is the surface $z = x^2$ parametrized by $\mathbf{x}(u, v) = (u, v, u^2)$. The tangent plane at $\mathbf{p}$, the origin, is the xy plane. However, if

$$\mathbf{q} = (2t, 0, 4t^2), \qquad \mathbf{r} = (t, t^2, t^2)$$

then the plane determined by $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ tends (as $t \to 0$) to the plane orthogonal to $(0, 1, 1)$.

8.4 Surface Integrals and Stokes' Theorem

Suppose that f is a continuous function defined in a domain D in R^3, and Σ is a surface in D. We can verify by an argument identical to that in Proposition 6 that the following definition makes sense independently of the coordinate choices involved.

Definition 9. Partition Σ into subsets $\Sigma_1, \ldots, \Sigma_n$ of surface patches on Σ. Define the **integral of f over Σ** to be

$$\int_\Sigma f\,dS = \sum_{i=1}^{n} \int_{\Sigma_i} fH\,du\,dv$$

where $H\,du\,dv$ is the surface area element in the patch containing Σ_i.

Examples

25. $I = \int_\Sigma x^2y^2z\,dS$, where Σ is the hemisphere Σ: $\{(x, y, z)$: $x^2 + y^2 + z^2 = 1, z > 0\}$. Using the same parametrization as in Example 23, we have

$$I = \int_{-\pi}^{\pi}\int_0^{\pi/2} \cos^5 u \cos^2 v \sin^2 v \sin u\,du\,dv$$

$$= \frac{1}{4}\int_{-\pi}^{\pi} \sin^2 2v\,dv \int_0^{\pi/2} \cos^5 u \sin u\,du = \frac{\pi}{24}$$

26. $I = \int_\Sigma(x + y^2)\,dS$, where Σ is the piece of the paraboloid given in Example 24.

$$I = 2\int_0^{2\pi}\int_0^1 [r^2 \cos u + r^3 \sin^2 u]\,dr\,du = \frac{\pi}{2}$$

Normal and Orientation

Let Σ be a surface in R^3. The tangent plane to Σ at a point $\mathbf{x}_0$ is a two-dimensional plane, thus its orthogonal complement is a line, called the **normal line** to Σ **at $\mathbf{x}_0$**. The **normal vector** N is a choice of unit vector lying on this line which varies continuously with the point. Such a choice is always possible locally, but is not always possible over the whole surface.

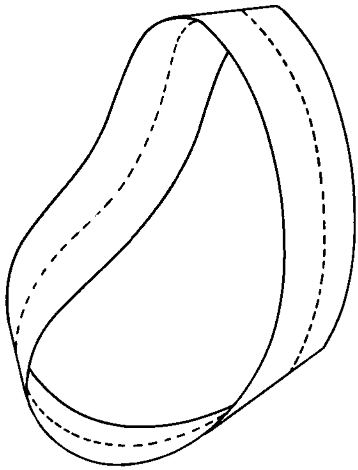

Figure 8.13

Consider the surface depicted in Figure 8.13 (called the Moebius band). This is obtained from a rectangle (Figure 8.14) by gluing together the vertical sides so that vertices with corresponding labels abut. There is no way to continuously select a normal vector to this surface which does not point in the opposite direction when traced around the circle in Figure 8.13. Notice that the same kind of phenomenon is put in evidence by Figure 8.14: a right-handed basis gets transformed into a left-handed basis when we cross the vertical line. We express this by saying that the Moebius band is **not orientable**.

Thus in two dimensions we find a problem which does not exist in one dimension. We ran into the same problem in the discussion of integration under a change of variable in the plane, and we successfully sidestepped it then. But we cannot avoid it now. We shall refer to an **orientation** on a surface Σ in R^3 as a choice of a sense of positive rotation in the tangent plane at every point. This choice is assumed to vary continuously: that is, if $\mathbf{v}_1, \mathbf{v}_2$ are nowhere collinear continuous vector fields defined on the surface and the rotation $\mathbf{v}_1 \to \mathbf{v}_2$ is positive at $\mathbf{x}_0$ it must be so in a neighborhood of $\mathbf{x}_0$. A choice of orientation is equivalent to a choice of normal vector. For, if a normal $\mathbf{N}$ is chosen we defined positive rotation in the tangent plane as follows: $\mathbf{v}_1 \to \mathbf{v}_2$ is positive if $\mathbf{v}_1 \to \mathbf{v}_2 \to \mathbf{N}$ is a right-handed system. Conversely, if an orientation is chosen we can define $\mathbf{N} = \mathbf{v}_1 \times \mathbf{v}_2$, where $\mathbf{v}_1, \mathbf{v}_2$ are unit vectors and the rotation $\mathbf{v}_1 \to \mathbf{v}_2$ is positive. If Σ is oriented and (u, v) are coordinates on a patch in Σ, we shall say that (u, v) is a **positively**

oriented coordinate system if the rotation $\mathbf{x}_u \to \mathbf{x}_v$ is positive. Here is a fact relating positively oriented coordinate systems which completes the discussion.

Proposition 7. *If (u, v) and (u', v') are two positively oriented coordinate systems defined on the oriented surface Σ, then*

$$\frac{\partial(u, v)}{\partial(u', v')} > 0$$

Examples

27. If f is a C^1 function defined on a domain D in the xy plane, then the graph

$$\Gamma(f): z = f(x, y)$$

is a surface patch. We consider it oriented so that the rotation from $\mathbf{x}_x = (1, 0, \partial f/\partial x)$ to $\mathbf{x}_y = (0, 1, \partial f/\partial y)$ is positive. Then the normal vector $\mathbf{N}$ always points upward out of the surface ($N^3 > 0$):

$$\mathbf{N} = \left[1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]^{-1}\left(-\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, 1\right)$$

28. More generally, we can always orient a surface patch $\Sigma: \mathbf{x} = \mathbf{x}(u, v)$, $(u, v) \in D$ by transferring the orientation from the u, v plane. That is, we take $\mathbf{x}_u \to \mathbf{x}_v$ as the positive sense of orientation. Then the normal to Σ is

$$\mathbf{N} = \|\mathbf{x}_u \times \mathbf{x}_v\|^{-1}(\mathbf{x}_u \times \mathbf{x}_v)$$

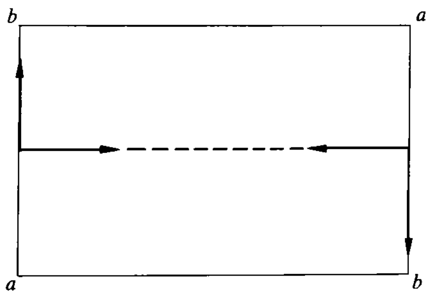

Figure 8.14

Just as, in the case of curves, we introduced the "vector length element" $d\mathbf{x} = \mathbf{T}ds$ along the curve, we introduce the vector area element $d\mathbf{S} = \mathbf{N}ds$ on a surface. Notice that, in terms of coordinates

$$d\mathbf{S} = \|\mathbf{x}_u \times \mathbf{x}_v\| \, du \, dv = \mathbf{x}_u \, du \times \mathbf{x}_v \, dv$$

(i.e., it is the vector product of the length elements along the parametric curves). In this way we can integrate vector fields along oriented surfaces:

Definition 10. If Σ is an oriented surface and $\mathbf{v}$ is a vector field defined around Σ, define the **flux of v across** Σ by

$$\int \langle \mathbf{v}, d\mathbf{S} \rangle = \int \langle \mathbf{v}, \mathbf{N} \rangle \, dS$$

The significance of the word flux will become apparent in the next section.

Example

29. Compute the flux of $\mathbf{v}(x, y, z) = (xy, yz, zx)$ across the graph of

$$f(x, y) = x^2 + 2y^2 \qquad x^2 + y^2 \leq 1$$

We take x, y as coordinates on Σ. Then

$$\int_\Sigma \langle \mathbf{v}, \mathbf{N} \rangle \, dS = \int_{x^2+y^2\leq 1} \left\langle \mathbf{v}, \frac{\partial \mathbf{x}}{\partial x} \times \frac{\partial \mathbf{x}}{\partial y} \right\rangle dx \, dy$$

$$= \int_{x^2+y^2\leq 1} \det\begin{pmatrix} xy & y(x^2+2y^2) & (x^2+2y^2)x \\ 1 & 0 & 2x \\ 0 & 1 & 4y \end{pmatrix} dx \, dy$$

$$= \int_{x^2+y^2\leq 1} [x^3 + 2y^2x - 2x^2y - 4x^2y^2 - 8y^3] \, dx \, dy$$

$$= 4 \int_0^{2\pi} \int_0^1 r^5 \cos^2\theta \sin^2\theta \, dr \, d\theta = \frac{\pi}{6}$$

Suppose that $\mathbf{v}$ is the velocity field of a flow and C is a closed path (oriented closed curve). In Section 8.2 we defined the circulation around a circle; we could use the same definition to define the **circulation around** C:

$$\text{circ}\,(C) = \int_C \langle \mathbf{v}, \mathbf{T} \rangle \, ds \tag{8.46}$$

($s =$ arc length along C, and $\mathbf{T}$ is the tangent vector to C). In Section 8.2 we used this idea to define curl $\mathbf{v}$, the "infinitesimal circulation" about a point; now we ask if we can recapture the total circulation from the infinitesimal. A clue is obtained by recognizing the integrand of (8.46) as the differential form associated to $\mathbf{v}$. If $\mathbf{v} = (v^1, v^2, v^3)$, then, on the curve $\langle \mathbf{v}, \mathbf{T} \rangle \, ds = \Sigma v^i \, dx^i = \langle \mathbf{v}, d\mathbf{x} \rangle$. What we are then asking for is the analog for surfaces of Green's theorem. Since the curl plays the same role in three variables that $d\omega$ plays in two, it is no accident that such a theorem exists.

Stokes' Theorem

Suppose now that Σ is an oriented surface lying in the domain of the vector field $\mathbf{v}$, and D is a subset of Σ bounded by a curve Γ. For the purposes of integration we must choose an orientation of Γ. It will be the natural one corresponding to the given orientation of Σ: Γ winds counterclockwise around D. To be more precise, we shall define the positively directed tangent. Let $\mathbf{p} \in \Gamma$ and consider a small path γ, with tangent vector $\mathbf{t}$ at $\mathbf{p}$ which crosses Γ and is directed so that it *enters* D. Then the tangent vector we wish to choose is that one $\mathbf{T}$ such that the rotation $\mathbf{T} \to \mathbf{t}$ is positive (see Figure 8.15). This corresponds to the counterclockwise sense of rotation about the normal to the tangent plane. When the boundary of D is so oriented it is a path, denoted ∂D. Now the theorem we have in mind (Stokes' theorem) asserts that the circulation around ∂D is given by

$$\int_{\partial D} \langle \operatorname{curl} \mathbf{v}, \mathbf{N} \rangle \, dS$$

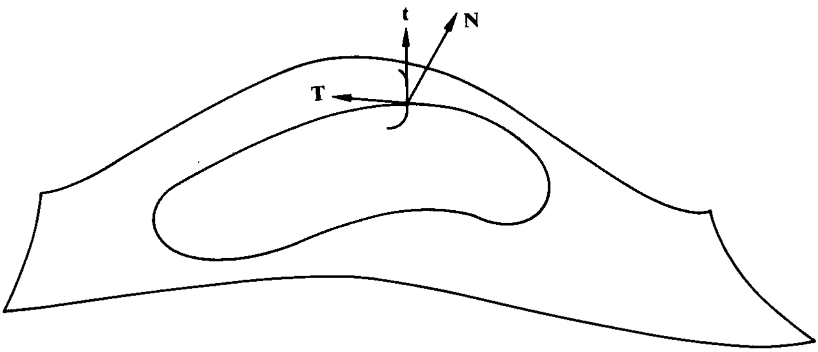

Figure 8.15

In order to derive this theorem from Green's theorem we must ensure that the conditions of Green's theorem will be met. Hence the following notion of a regular domain.

Definition 11. Let Σ be a surface in R^3. A subset D of Σ will be called a **regular domain** if it can be partitioned into finitely many subsets of surface patches which correspond to regular domains in the plane in the particular coordinate representation.

Theorem 8.3. *Let* $\mathbf{v}$ *be a vector field defined in a domain* U *in* R^3, *and suppose* Σ *is an oriented surface lying in* U *with normal* $\mathbf{N}$ *in* U. *Let* D *be a regular domain in* Σ *whose boundary* ∂D *is a curve. Then*

$$\int_{\partial D} \langle \mathbf{v}, \mathbf{T} \rangle \, ds = \int_D \langle \operatorname{curl} \mathbf{v}, \mathbf{N} \rangle \, dS \tag{8.47}$$

Proof. Since D is regular, there are coordinate patches $\Sigma_1, \ldots, \Sigma_n$ and a partition $D = D_1 \cup \cdots \cup D_n$ of D such that $D_i \subset \Sigma_i$ and D_i corresponds to a regular domain in the Σ_i coordinates. Now let $B_1, \ldots, B_m$ be balls in R^3 such that $D \subset B_1 \cup \cdots \cup B_n$ and each B_j lies completely inside one of the coordinate patches Σ_i. Let $\rho_1, \ldots, \rho_n$ be a partition of unity subordinate to this cover. Then, since $\Sigma \rho_j = 1$ on D,

$$\int_{\partial D} \langle \mathbf{v}, \mathbf{T} \rangle \, ds = \sum_j \int_{\partial D} \langle \rho_j \mathbf{v}, \mathbf{T} \rangle \, ds = \sum_{i,j} \int_{\partial(D_i)} \langle \rho_j \mathbf{v}, \mathbf{T} \rangle \, ds$$

since each part of ∂D_i which is not on ∂D appears as part of ∂D_j for some $j \neq i$ and with the opposite orientation.

$$\int_D \langle \operatorname{curl} \mathbf{v}, \mathbf{N} \rangle \, dS = \sum_j \int_D \langle \operatorname{curl}(\rho_j \mathbf{v}), \mathbf{N} \rangle \, dS = \sum_{i,j} \int_{D_i} \langle \operatorname{curl}(\rho_j \mathbf{v}), \mathbf{N} \rangle \, dS$$

Thus, we only need to show that the right-hand sides are equal termwise: we may assume that we are in a coordinate patch.

This is now our situation. Let Σ be a surface patch coordinatized by

$$\mathbf{x} = (x^1(u, v), x^2(u, v), x^3(u, v)), \; (u, v) \in N \subset R^2$$

and suppose Δ is a regular domain in N and D is the subdomain of Σ corresponding to Δ: $D = \{\mathbf{x}(u, v), (u, v) \in N\}$. Let $\mathbf{v} = (v^1, v^2, v^3)$ be a vector field defined on Σ. Then we must verify

$$\int_{\partial D} \langle \mathbf{v}, \mathbf{T} \rangle \, ds = \int_D \langle \operatorname{curl} \mathbf{v}, \mathbf{N} \rangle \, dS$$

This is just the computation that $\langle \operatorname{curl} \mathbf{v}, \mathbf{N}\rangle \, dS = (d\omega_v)\, du\, dv$ under the change of variables. First, we study the left integral:

$$\int_{\partial D} \langle \mathbf{v}, \mathbf{T}\rangle \, ds = \int_{\partial D} \Sigma v^i \, dx^i = \int_{\partial \Delta} \Sigma v^i \frac{\partial x^i}{\partial u} du + \Sigma v^i \frac{\partial x^i}{\partial v} dv$$

By Green's theorem this is

$$\sum_i \int_\Delta \left[\frac{\partial}{\partial u}\left(v^i \frac{\partial x^i}{\partial v}\right) - \frac{\partial}{\partial v}\left(v^i \frac{\partial x^i}{\partial u}\right)\right] du\, dv \tag{8.48}$$

Now

$$\frac{\partial}{\partial u}\left(v^i \frac{\partial x^i}{\partial v}\right) = \sum_j \frac{\partial v^j}{\partial x^j} \frac{\partial x^j}{\partial u} \frac{\partial x^i}{\partial v} + v^i \frac{\partial^2 x^i}{\partial u\, \partial v}$$

$$\frac{\partial}{\partial v}\left(v^i \frac{\partial x^i}{\partial v}\right) = \sum_j \frac{\partial v^i}{\partial x^j} \frac{\partial x^j}{\partial v} \frac{\partial x^i}{\partial u} + v^i \frac{\partial^2 x^i}{\partial v\, \partial u}$$

The integrand in (8.48) is thus

$$\begin{aligned}
&\sum_{i,j} \frac{\partial v^i}{\partial x^j}\left(\frac{\partial x^j}{\partial u}\frac{\partial x^i}{\partial v} - \frac{\partial x^j}{\partial v}\frac{\partial x^i}{\partial u}\right) \\
&= \sum_{i<j} \left(\frac{\partial v^i}{\partial x^j} - \frac{\partial v^j}{\partial x^i}\right)\left(\frac{\partial x^j}{\partial u}\frac{\partial x^i}{\partial v} - \frac{\partial x^j}{\partial v}\frac{\partial x^i}{\partial u}\right) \\
&= \langle \operatorname{curl} \mathbf{v}, \mathbf{x}_u \times \mathbf{x}_v \rangle
\end{aligned}$$

Hence, after Green's theorem the left integral becomes

$$\int_\Delta \langle \operatorname{curl} \mathbf{v}, \mathbf{x}_u \times \mathbf{x}_v \rangle \, du\, dv$$

But the right integral is

$$\int_\Delta \langle \operatorname{curl} \mathbf{v}, \mathbf{N}\rangle \|\mathbf{x}_u \times \mathbf{x}_v\| \, du\, dv = \int_\Delta \langle \operatorname{curl} \mathbf{v}, \mathbf{x}_v \times \mathbf{x}_v \rangle \, du\, dv$$

since $\mathbf{x}_u \times \mathbf{x}_v = \|\mathbf{x}_u \times \mathbf{x}_v\| \mathbf{N}$. The proof is concluded.

Examples

30. Calculate $\int_{\partial \Sigma} \langle \mathbf{v}, d\mathbf{x}\rangle$, where Σ is the surface

$$\Sigma\colon z = x^2 \qquad 0 \le x \le 1 \qquad 0 \le y \le 1$$

and $\mathbf{v}(x, y, z) = -(y, z, x)$. We make the computations:

$$\operatorname{curl} \mathbf{v} = -(1, 1, 1)$$
$$\mathbf{x}_x = (1, 0, 2x)$$
$$\mathbf{x}_y = (0, 1, 0)$$
$$d\mathbf{S} = (\mathbf{x}_x \times \mathbf{x}_y)\, dx\, dy = (-2x, 0, 1)\, dx\, dy$$

$$\int_{\partial\Sigma} \langle \mathbf{v}, d\mathbf{x}\rangle + \int_{\Sigma} \langle \operatorname{curl} \mathbf{v}, d\mathbf{S}\rangle = \int_0^1 \int_0^1 (2x + 1)\, dx\, dy = 0$$

31. Let Σ be the surface patch

$$\mathbf{x} = \mathbf{x}(u, v) = (u \cos v, u \sin v, u \cos 6v) \qquad 0 \leq u \leq 1 \quad 0 \leq v \leq 2\pi$$

Let $\mathbf{N} = (N^1, N^2, N^3)$ be the normal to Σ. Then

$$\int_{\Sigma} (N^1 + N^2 + N^3)\, dS = \int_{\Sigma} \langle \operatorname{curl} \mathbf{v}, dS\rangle$$

where $\mathbf{v} = (y, z, x)$. Thus, the sought-for integral can be computed

as $\int_{\gamma} \langle \mathbf{v}, d\mathbf{x}\rangle$, where γ is the curve $u = 1$:

$$\gamma\colon \mathbf{x} = \mathbf{x}(v) = (\cos v, \sin v, \cos 6v)$$

$$\int_{\gamma} \langle \mathbf{v}, d\mathbf{x}\rangle = \int_0^{-2\pi} (-\sin^2 v + \cos v \cos 6v - 6 \cos v \sin 6v)\, dv = -\pi$$

• EXERCISES

19. Calculate $\int_{\Sigma} f\, dS$, where
 (a) $f(x, y, z) = x^2 + 2y \qquad \Sigma\colon z^2 = x^2 + y^2 \qquad 0 \leq z \leq 1$
 (b) $f(x, y, z) = xy + yz + zx \qquad \Sigma\colon z = x^2 + y^2 \qquad 0 \leq z \leq 1$
 (c) $f(x, y, z) = xyz \qquad \Sigma\colon \mathbf{x}(u, v) = (u \cos u, u \sin v, v \sin u)$
$0 \leq u \leq 2\pi \qquad 0 \leq v \leq 2\pi$
20. Calculate $\int \langle \mathbf{v}, d\mathbf{S}\rangle$, where
 (a) $\mathbf{v}(x, y, z) = (xy, yz, zx) \qquad \Sigma\colon z = e^{xy} \qquad 0 \leq x \leq 1$
$0 \leq y \leq 1$
 (b) $\mathbf{v}(x, y, z) = (1, -y, x) \qquad \Sigma\colon x^2 + y^2 + z^2 = 1$
 (c) $\mathbf{v}(x, y, z) = (1, 0, y) \qquad \Sigma\colon z = x^2 - y^2 \qquad x^2 + y^2 \leq 1$

21. Suppose $\mathbf{v}$ is a vector field defined in a neighborhood of the domain D. Show that

$$\int_{\partial D} \langle \text{curl } \mathbf{v}, d\mathbf{S} \rangle = 0$$

22. (a) Suppose that D is a regular domain on a surface Σ. Verify that for any vector $\mathbf{a}$,

$$\frac{1}{3}\int_{\partial D} \langle \mathbf{a} \times \mathbf{x}, d\mathbf{x} \rangle = \int_D \langle \mathbf{a}, d\mathbf{S} \rangle$$

(b) Just as we integrated vector functions on the interval, we can integrate vector functions on lines and surfaces (and in space). Show that, for a regular domain D these vectors are the same:

$$\int_D d\mathbf{S} = \frac{1}{3}\int_{\partial D} \mathbf{x} \times d\mathbf{x}$$

(*Hint*: This follows from part (a).)

23. Show that if u, v are C^1 functions on the regular domain D that

$$\int_{\partial D} \langle u\nabla v, d\mathbf{x} \rangle = \int_D \langle \nabla u \times \nabla v, d\mathbf{S} \rangle$$

• PROBLEMS

29. If ω is a closed form defined in a neighborhood of the unit sphere in R^3, show that there is a function f such that $\omega = df$ *on* the sphere.

30. Consider the torus T:

$\mathbf{x} = \mathbf{x}(u, v) = 2 \cos u + \cos v$

$\mathbf{x} = \mathbf{x}(u, v) = (2 + \cos v)\cos u, (2 + \cos v)\sin u, \sin v)$

(a) Show that the differentials du, dv are well-defined differential forms on T.

(b) If ω is a closed form defined on T, show that the integrals

$$\int_\Gamma \omega \qquad \int_\gamma \omega$$

are constant as Γ ranges over all circles $v =$ constant, and γ ranges over all circles $u =$ constant.

(c) If ω is a closed form there are constants c_1, c_2 and a differentiable function f such that

$$\omega = c_1\, du + c_2\, dv + df$$

(*Hint*: Take $c_1 = \int_\Gamma \omega/2\pi$, $c_2 = \int_\gamma \omega/2\pi$, where the integrals are taken as defined in part (b).)

31. State and prove a fact like that in Problem 30(c) when T is replaced by a cylinder.

32. Verify this restatement of Stokes' theorem: Let $\mathbf{v} = (F, G, H)$ be a vector field defined in a domain U in R^3 and suppose that Σ is an oriented surface lying in U with normal $\mathbf{N} = (\cos\alpha, \cos\beta, \cos\gamma)$. If D is a regular domain in Σ, then

$$\int_{\partial D} F\,dx + G\,dy + H\,dz$$

$$= \int_D \left[\left(\frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}\right)\cos\alpha + \left(\frac{\partial F}{\partial z} - \frac{\partial H}{\partial x}\right)\cos\beta + \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right)\cos\gamma\right] dS$$

33. Let D be a regular domain on the oriented surface Σ. Show that if $\mathbf{v}$ is a vector field defined on Σ

$$\int_{\partial D} \langle \mathbf{v}, \mathbf{N}_\gamma \rangle\, dS = \int_D \langle \operatorname{curl}(\mathbf{v} \times \mathbf{N}), \mathbf{N} \rangle\, dS$$

where $\mathbf{N}_\gamma$ is the unit surface normal to ∂D (see Problem 24).

8.5 The Divergence Theorem

Let $\mathbf{v}$ be a vector field defined in a domain $U \subset R^3$, and $\mathbf{x} = \boldsymbol{\phi}(\mathbf{x}_0, t)$ the associated steady flow. Let D be a domain whose closure is contained in U such that ∂D is sufficiently differentiable surface. Notice that ∂D is orientable, since we can choose as normal vector the unit vector $\mathbf{N}$ which is exterior to the domain D. We shall assume throughout this section that this is the chosen normal. For a small interval of time Δt, let us attempt to calculate the amount of fluid that passes through ∂D. For $\mathbf{x}_0 \in D$, the particle at the point $\boldsymbol{\phi}(\mathbf{x}_0, -t)$ at time 0 for $0 \le t \le \Delta t$ passes through $\mathbf{x}_0$, since $\boldsymbol{\phi}(\mathbf{x}_0, -t + t) = \boldsymbol{\phi}(\mathbf{x}_0, 0) = \mathbf{x}_0$. Thus, the volume of the fluid passing through ∂D at time Δt is the volume of the domain

$$D_{\Delta t} = \{\mathbf{x} \colon \mathbf{x} = \boldsymbol{\phi}(\mathbf{x}_0, -t) \colon 0 \le t \le \Delta t, \mathbf{x}_0 \in \partial D\}$$

We shall approximate this volume by linearizing locally. That is, we cover ∂D by small neighborhoods U_i, and replace $U_i \cap \partial D$ by the piece T_i of the tangent plane to ∂D with the same area at some point in U_i. We assume also that ϕ is a pure translation through T_i. Then the volume which passes through T_i is a parallelepiped of volume

$$\langle \phi(\mathbf{x}_i, -\Delta t) - \phi(\mathbf{x}_i, 0), \mathbf{N} \rangle \, \Delta A_i$$

where $\mathbf{x}_i$ is some point in $U_i \cap \partial D$, and ΔA_i is the area of T_i. Let us point out that this is a signed volume; the sign being positive if the flow is into D (since N is the exterior normal, and if $\phi(\mathbf{x}_i, -\Delta t)$ is on the same side of ∂D as N, $\langle \phi(\mathbf{x}_i, -\Delta t) - \phi(\mathbf{x}_i, 0), \mathbf{N} \rangle$ is positive). This is in fact what we want, for we want to discover the flow *into* D rather that the flow *through* ∂D.

It follows that an approximation to the volume of $D_{\Delta t}$ is

$$\sum_i \langle \phi(\mathbf{x}_i, -\Delta t) - \phi(\mathbf{x}_i, 0), \mathbf{N} \rangle \, \Delta A_i$$

and by letting the covering get arbitrarily fine, we may replace this by an integral:

$$\int_{\partial D} \langle \phi(\mathbf{x}, -\Delta t) - \phi(\mathbf{x}, 0), \mathbf{N} \rangle \, ds \tag{8.49}$$

The limit of $1/\Delta t$ times (8.49) as $\Delta t \to 0$ through positive values is the instantaneous flow into D, or the **flux into** D at time $t = 0$.

Proposition 8. *The flux out of D at time $t = 0$ is*

$$\int \int_{\partial D} \langle \mathbf{v}, d\mathbf{S} \rangle$$

Proof. The flux out of D is

$$\begin{aligned} &-\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{\partial D} \langle \phi(\mathbf{x}, -\Delta t) - \phi(\mathbf{x}, 0), \mathbf{N} \rangle \, dS \\ &= \lim_{\Delta t \to 0} \frac{1}{-\Delta t} \int_{\partial D} \langle \phi(\mathbf{x}, -\Delta t) - \phi(\mathbf{x}, 0), d\mathbf{S} \rangle \\ &= \int_{\partial D} \langle \lim_{\Delta t \to 0} \frac{1}{\Delta t} [\phi(\mathbf{x}, \Delta t) - \phi(\mathbf{x}, 0)], d\mathbf{S} \rangle = \int_{\partial D} \langle \mathbf{v}, d\mathbf{S} \rangle \end{aligned}$$

Now the flux out of D is the instantaneous rate of flow of fluid out of D. On physical grounds this should be identical to the instantaneous rate of expansion of the fluid in D, which is (as in Section 8.1) $\int_D \operatorname{div} \mathbf{v} \, dV$. Thus, we should expect

$$\iint_D \langle \mathbf{v}, d\mathbf{S} \rangle = \iiint_D \operatorname{div} \mathbf{v} \, dV \tag{8.50}$$

and in fact this is the case. Equation (8.50) is known as the **divergence theorem**. For suitable domains it is an easy consequence of the fundamental theorem of calculus. As in the case of Green's theorem, we shall call such domains, or finite unions of such domains, **regular domains**. Many domains in R^3 are regular, but by no means are all regular. The general theorem, for an arbitrary domain, is not easy to prove and we shall here avoid the issue.

Definition 12. A domain D in R^3 is **regular** if it can be expressed in each of these ways:

$$\begin{aligned} D &= \{(x, y, z) : (x, y) \in D_1 \quad f(x, y) \le z \le g(x, y)\} \\ &= \{(x, y, z) : (x, z) \in D_2 \quad r(x, z) \le y \le s(x, z)\} \\ &= \{(x, y, z) : (y, z) \in D_3 \quad u(y, z) \le x \le v(y, z)\} \end{aligned}$$

where all functions are continuously differentiable.

Lemma. If $\mathbf{v}$ is a differentiable vector field defined in a neighborhood of the regular domain D, then

$$\int_{\partial D} \langle \mathbf{v}, d\mathbf{S} \rangle = \int_D \operatorname{div} \mathbf{v} \, dV$$

Proof. Let $\mathbf{v} = v^1\mathbf{E}_1 + v^2\mathbf{E}_2 + v^3\mathbf{E}_3$.

$$\operatorname{div} \mathbf{v} = \frac{\partial v^1}{\partial x^1} + \frac{\partial v^2}{\partial x^2} + \frac{\partial v^3}{\partial x^3}$$

We shall show that for each i,

$$\int_{\partial D} \langle v^i\mathbf{E}_i, d\mathbf{S} \rangle = \int_D \frac{\partial v^i}{\partial x^i} \, dV$$

Then the lemma will follow by summing over i. To prove the ith case, we use the appropriate representation of the domain. Since all cases are then the same, we shall only verify one case, say the third.

Now, using the expression

$$D = \{(x, y, z) : (x, y) \in D_1, f(x, y) \le z \le g(x, y)\}$$

the boundary of D consists of the part Σ_0 lying over ∂D_1 and the two surfaces

$$\Sigma_1: z = f(x, y) \qquad (x, y) \in D_1$$
$$\Sigma_2: z = g(x, y) \qquad (x, y) \in D_1$$

Since $\mathbf{E}_3$ is tangent to the surface lying over ∂D_1 at every point, the left-hand integral over Σ_0 vanishes.

Now Σ_1 has the parametrization

$$\mathbf{x} = (x, y, f(x, y)) \qquad (x, y) \in D_1$$

Since the domain lies above this surface, the exterior normal points downward, so is determined by $-\mathbf{x}_x \times \mathbf{x}_y$ (see Figure 8.16). Now

$$\mathbf{x}_x = (1, 0, f_x) \qquad \mathbf{x}_y = (0, 1, f_y)$$

so we have $d\mathbf{S} = (f_y, f_x, -1)\, dx\, dy$. Then

$$\int_{\Sigma_1} \langle v^3\mathbf{E}_3, d\mathbf{S}\rangle = -\int_{D_1} v^3(x, y, f(x, y))\, dx\, dy$$

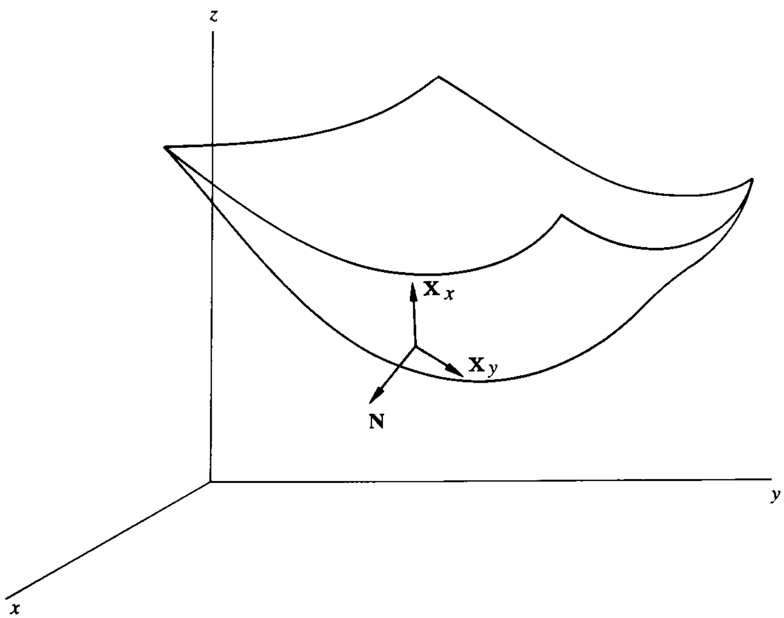

Figure 8.16

A similar computation produces

$$\int_{\Sigma_2} \langle v^3 \mathbf{E}_3, d\mathbf{S} \rangle = \int_{D_1} v^3(x, y, g(x, y))\, dx\, dy$$

Now, we compute $\int_D (\partial v^3/\partial z)\, dV$ by Fubini's theorem.

$$\int_D \frac{\partial v^3}{\partial z}\, dV = \int_{D_1} \left[\int_{f(x,y)}^{g(x,y)} \frac{\partial v^3}{\partial z}(x, y, z)\, dz \right] dx\, dy$$

$$= \int_{D_1} [v^3(x, y, g(x, y)) - v^3(x, y, f(x, y))]\, dx\, dy$$

by the fundamental theorem of calculus. But this is, according to our previous calculations the same as $\int_{\partial D} \langle v^3 \mathbf{E}_3, d\mathbf{S} \rangle$. Thus the lemma is verified.

Theorem 8.4. (Divergence Theorem) *Let* $\mathbf{v}$ *be a continuously differentiable vector field defined in a domain* D *in* R^3. *Suppose* D *can be covered by finitely many balls* $B_1, \ldots, B_n$ *such that each* $D \cap B_i$ *is a regular domain. Then*

$$\int_{\partial D} \langle \mathbf{v}, d\mathbf{S} \rangle = \int_D \operatorname{div} \mathbf{v}\, dV$$

Proof. Let $\rho_1, \ldots, \rho_n$ be a partition of unity subordinate to $B_1, \ldots, B_n$. Then

$$\int_{\partial D} \langle \mathbf{v}, d\mathbf{S} \rangle = \sum_i \int_{\partial D} \langle \rho_i \mathbf{v}, d\mathbf{S} \rangle = \sum \int_{\partial(D \cap B_i)} \langle \rho_i \mathbf{v}, d\mathbf{S} \rangle$$

$$\int_D \operatorname{div} \mathbf{v}\, dV = \sum_i \int_D \operatorname{div}(\rho_i \mathbf{v})\, dV = \sum_i \int_{D \cap B_i} \operatorname{div}(\rho_i \mathbf{v})\, dV$$

for the customary reasons: $\Sigma \rho_i = 1$ and $\rho_i = 0$ outside B_i. By the lemma, the right-hand sides are the same termwise, so the left-hand sides are the same. We shall henceforth describe domains of the type referred to in Theorem 8.4 as **regular**.

Examples

32. First of all, the result of Exercise 22 follows easily from the divergence theorem, since div curl $\mathbf{v} = 0$. For then

$$\int_{\partial D} \langle \operatorname{curl} \mathbf{v}, d\mathbf{S} \rangle = \int_D \operatorname{div} \operatorname{curl} \mathbf{v}\, dV = 0$$

33. Let

$$D = \{x^2 + y^2 + z^2 \le 1\}, f(x, y, z) = x^2 + y^2 + z^2$$

Then

$$\int_{\partial D} \langle \nabla f, d\mathbf{S} \rangle = \int_D \operatorname{div} \nabla f \, dV = 6 \int_D dV = 8\pi$$

34. Let D be the domain $\{1 \ge z \ge x^2 + y^2\}$, and let $\mathbf{v}(x, y, z) = (xy, yz, x)$. Then $\operatorname{div} \mathbf{v} = y + z$ and

$$\begin{aligned}
\int_D \operatorname{div} \mathbf{v} \, dV &= \int_0^1 \left[\int_{x^2+y^2 \le z} (y + z) \, dx \, dy \right] dz \\
&= \pi \int_0^1 z^2 \, dz = \frac{\pi}{3}
\end{aligned}$$

$$\begin{aligned}
\int_{\partial D} \langle \mathbf{v}, d\mathbf{S} \rangle &= \int_{z=1} \langle \mathbf{v}, d\mathbf{S} \rangle - \int_{z=x^2+y^2} \langle \mathbf{v}, d\mathbf{S} \rangle \\
&= \int_{x^2+y^2 \le 1} x \, dx \, dy - \int_{x^2+y^2 \le 1} \langle (xy, y(x^2 + y^2), x), \\
&\qquad (-2x, 2y, 1) \rangle \, dx \, dy \\
&= 2 \int_{x^2+y^2 \le 1} (y^2 x^2 + y^4) \, dx \, dy = \frac{\pi}{3}
\end{aligned}$$

The Heat Equation

In Chapter 6 in our discussion of the heat equation we postponed its derivation in dimensions greater than one. We had to await the divergence theorem; with that we can carry through our argument just as in the one-dimensional case. Thus, we suppose a homogeneous metallic object U in R^3 has at time t a temperature distribution $u(\mathbf{x}, t)$. According to the laws of thermodynamics, the vector field $\mathbf{q}$ associated to the flow of heat energy is proportional to the gradient of the temperature, but for sign:

$$\mathbf{q} + c \nabla u = 0 \tag{8.51}$$

Another basic principle is this: The increase in temperature of a unit mass is proportional to the increase in heat energy. More specifically, the change in energy in any given domain D in a time interval t is given by

$$k\rho \int_D \Delta u \, dV$$

where $\Delta u(\mathbf{x}, \Delta t)$ is the change in temperature at $\mathbf{x}$ over the period Δt, ρ is the density, and k is the proportionality constant (the specific heat). Thus, the rate of increase of heat energy in D is

$$k\rho \int_D \frac{\partial u}{\partial t} \, dV$$

Now, we can compute (using the law of conservation of energy) the rate of increase of energy in D; it is the flux into D across the boundary. Thus we obtain this basic equation for every domain D:

$$-\int_{\partial D} \langle \mathbf{q}, d\mathbf{S} \rangle = k\rho \int_D \frac{\partial u}{\partial t} \, dV$$

By the divergence theorem and (8.51) we have

$$\int_D \operatorname{div} \nabla u \, dV = \frac{k\rho}{c} \int_D \frac{\partial u}{\partial t} \, dV$$

for every domain D. Thus the two functions must be the same, and we obtain the heat equation:

$$\operatorname{div} \nabla u = \left(\frac{k\rho}{c}\right) \frac{\partial u}{\partial t}$$

As we saw in Chapter 6, the steady state (or equilibrium) temperature distribution solves Laplace's equation:

$$\operatorname{div} \nabla u = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

• EXERCISES

24. If Σ is an oriented surface with normal $\mathbf{N}$ and f is a C^1 function defined near Σ, we denote $\langle \nabla f, \mathbf{N} \rangle$ by $\partial f / \partial \mathbf{N}$. Show that

$$\int_{\partial D} \frac{\partial f}{\partial \mathbf{N}} \, dS = \int_D \Delta f \, dV$$

for any regular domain D.

25. If $\mathbf{v}$ is a vector field such that div $\mathbf{v} = 1$, then for any regular domain D

$$\text{vol}(D) = \int_{\partial D} \langle \mathbf{v}, d\mathbf{S} \rangle$$

In particular, we may make any one of these choices for $\mathbf{v}$:

$$(x, 0, 0) \qquad (0, y, 0) \qquad (0, 0, z)$$

Find the volume of these domains, using the divergence theorem.

(a) The cap $z \geq ax^2 + by^2 \quad 0 \leq z \leq 3$.

(b) The cone $z^2 \geq ax^2 + by^2 \quad 0 \leq z \leq 1$.

(c) The tetrahedron bounded by the planes $z = 0 \quad x + y + z = 1$ $x = 2y$, $y = 0$.

26. Verify this formula for any regular domain:

$$4 \int_D \|\mathbf{x}\| \, dV = \int_{\partial D} \|\mathbf{x}\| \langle \mathbf{x}, d\mathbf{S} \rangle$$

27. Here is another way of expressing the divergence theorem, which is free of vector notation. Express N in terms of its direction cosines:

$$\mathrm{N} = (\cos \alpha, \cos \beta, \cos \gamma)$$

Then for any three functions F, G, H,

$$\int_{\partial D} (F \cos \alpha + G \cos \beta + H \cos \gamma) \, dS = \int_D \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} \right) dV$$

28. Compute

(a) $\int_\Sigma \langle (x^2, y^2, z^2), d\mathbf{S} \rangle$ where Σ is the (oriented) surface of the cube with side edge 2, and center at the origin.

(b) $\int (x \cos \alpha - y \cos \beta - z \cos \gamma) \, dS$ over the sphere S: $x^2 + y^2 + (z - 1)^2 = 1$, where $(\cos \alpha, \cos \beta, \cos \gamma)$ is the normal.

• PROBLEMS

34. Let Σ be a surface which intersects each ray from the origin in at most one point. The set of rays which intersect Σ will pierce the unit sphere in a set S. The area of S is the **solid angle subtended by** Σ. Show that the solid angle is given by

$$\int_\Sigma \frac{\langle \mathbf{x}, d\mathbf{S} \rangle}{\|\mathbf{x}\|^3}$$

35. Vector-valued functions can easily be integrated over any domain, coordinate by coordinate. Verify these formulas for a regular domain D:

$$\int_{\partial D} \mathbf{v} \times d\mathbf{S} = \int_D \operatorname{curl} \mathbf{v}\, dV$$

$$\int_{\partial D} f\, d\mathbf{S} = \int_D \nabla f\, dV$$

$$\int_{\partial D} \mathbf{N}\, dS = 0$$

36. Let $\mathbf{v}$ be a divergence-free vector field defined in a domain U. Show that if γ is a closed curve defined in U, then for any regular domain D on a surface Σ such that $\partial D = \gamma$, the integral

$$\int_D \langle \mathbf{v}, d\mathbf{S} \rangle$$

always has the same value.

37. Show that the function f is harmonic in the domain D if and only if, for every ball $B \subset D$,

$$\int_{\partial B} \langle \nabla f, d\mathbf{S} \rangle = 0$$

38. Suppose there is given a flow in R^3 with these properties:

(a) The flow has constant velocity outside of some large bounded set.

(b) The flow on the $\{z = 0\}$ plane remains along that plane (no fluid passes from the upper half space to the lower half space). Show that

$$\int_H \operatorname{div} \mathbf{v}\, dV = 0$$

where H is the half space $\{z \geq 0\}$.

8.6 Dirichlet's Principle

Let D be a domain in R^3, and suppose $\mathbf{v}$ is the velocity field of a flow through D which is steady (time independent). The total kinetic energy of the flow is given by the integral

$$\frac{1}{2} \int_D \rho \|\mathbf{v}\|^2\, dV \tag{8.52}$$

where ρ is the density of the fluid (we shall here take ρ to be constant). An important physical problem is this: find the flow which minimizes the energy (8.52) subject to certain conditions being fixed on ∂D. For example, we may assume that the normal component of the flow $\langle \mathbf{v}, \mathbf{N} \rangle$ through the boundary is fixed. Or we may assume that the flow is conservative, that is, $\mathbf{v}$ has a potential function, and the values of the potential are fixed on the boundary. These problems are analogous to Neumann's and Dirichlet's problems respectively (see Chapter 6). Dirichlet's principle is that the flow which minimizes the energy is the gradient of a harmonic function (solution of Laplace's equation). In this section we shall derive Dirichlet's principle and indicate how the techniques involved can be used to discover the solution to the problems. In order to do this, let us make these problems precise. Let D be a domain in R^3, and f a function defined on D.

I. **(Dirichlet's Problem)** Among all C^2 functions u defined on D which have the boundary values f, find the one which minimizes the integral

$$\int_D \|\nabla u\|^2 \, dV \tag{8.53}$$

II. **(Neumann's Problem)** Among all C^2 functions u defined on D such that $\langle \nabla u, \mathbf{N} \rangle = f$ on ∂D, find the one which minimizes the integral (8.53).

In order to study these problems we need (i) to relate boundary data to the integral (8.53), (ii) to discover an interpretation of (8.53) which will suggest a technique for minimizing that integral. The first need is filled by the divergence theorem, which will take the form of Green's identities (given below). The interpretation requested in (ii) is that of Euclidean vector spaces and the technique will be orthogonal projection. Let us describe this idea more fully.

Let $C^2(D)$ represent the collection of functions which are twice continuously differentiable on D. We can make this vector space into a Euclidean vector space by defining on it the inner product

$$E\langle u, v \rangle = \int_D \langle \nabla u, \nabla v \rangle \, dV \tag{8.54}$$

Then (8.53) is the square of the length of ∇u in terms of this inner product. We shall denote (8.53) by $E^2\langle u \rangle$. Our problem is to minimize this length among all functions with the given boundary value f. Let M_f be the space of functions in $C^2(D)$ with boundary value f. Then M_f is a translate of the space M_0: if u is a function with boundary value f, then $M_f = \{u + g : g \in M_0\}$. Now it is a simple principle of Euclidean vector spaces that the vector in M_f which is closest to 0 is orthogonal to M_f, hence also orthogonal to M_0.

The solution to our problem will then be that function in $M_f \cap M_0^\perp$. Finally, we can identify $M_0^\perp$ as the space of harmonic functions.

There is one fault with our reasoning. The "simple principle" above is one about finite-dimensional Euclidean vector spaces (recall Chapter 1), and it is not necessarily true in the infinite-dimensional case (of which ours is a prime example). The problem is that there need not be *any* point in $M_f \cap M_0^\perp$; and our argument will be complete once this problem of existence is resolved. The mid-19th century mathematicians such as Dirichlet and Riemann were little troubled by such problems; it was during the late 19th century that mathematicians began to think of existence questions as crucial (with good reason). And it was not until the last decade of that century that the existence problem was effectively solved. (The reader is referred to the history by Kellogg (pp. 277–286) for a fuller account.)

The link between the geometry described above and the subject of harmonic functions comes out of certain computations involving the divergence theorem (Green's identities). These will now be exposed. We shall adopt one more notational convention before proceeding (already foreseen in the problems): if u is defined on the oriented surface Σ, then $\langle \nabla u, \mathrm{N} \rangle$ is the directional derivative of u in the direction normal to Σ. We shall denote it by $\partial u/\partial N$.

Theorem 8.5. (Green's Identities) *Let f, g be two C^2 functions defined on a regular domain D. Then*

$$\int_{\partial D} f \frac{\partial q}{\partial N}\, dS = \int_D [f \Delta g + \langle \nabla f, \nabla g \rangle]\, dV \tag{8.55}$$

Proof.

$$\int_{\partial D} f \frac{\partial g}{\partial N}\, dS = \int_{\partial D} \langle f \nabla g, \mathrm{N} \rangle\, dS = \int_D \operatorname{div}(f \nabla g)\, dV$$

But, as is easily computed (see Exercise 10):

$$\operatorname{div}(f \nabla g) = f \operatorname{div} \nabla g + \langle \nabla f, \nabla g \rangle$$

so Theorem 8.5 is proven.

Corollary 1.

(i) *If g is harmonic, $\int_{\partial D} f(\partial g/\partial N)\, dS = E\langle f, g \rangle$.*
(ii) *If $f \in M_0$ and g is harmonic, $E\langle f, g \rangle = 0$.*

(iii) *If f and g are harmonic,*

$$\int_{\partial D} f \frac{\partial q}{\partial N}\, dS = \int_{\partial D} g \frac{\partial f}{\partial N}\, dS \tag{8.56}$$

(iv) *If f is orthogonal to every function in M_0, f is harmonic.*

Proof.
(i) If g is harmonic, then $\Delta g = 0$, so by (8.55) we have

$$\int_{\partial D} f \frac{\partial g}{\partial N}\, dS = \int_D \langle \nabla f, \nabla g \rangle\, dV = E\langle f, g \rangle \tag{8.57}$$

(ii) Now, if $f \in M_0$, f has boundary values 0, so the integral on the left of (8.57) also vanishes, and thus $E\langle f, g\rangle = 0$.
(iii) If g is harmonic, we have (8.57). If f is also harmonic we may interchange the roles of f and g in (8.57) obtaining

$$\int_{\partial D} g \frac{\partial f}{\partial N}\, dS = E\langle g, f\rangle$$

Thus (8.56) results since $E\langle g, f\rangle = E\langle f, g\rangle$.
(iv) If $g \in M_0$, then by (8.55) (interchanging f and g), we have

$$\int_D g\Delta f\, dV + E\langle f, g\rangle = 0$$

Now if f is orthogonal to M_0, then $\int g\Delta f\, dV = 0$ for every g with boundary value zero. This implies that $\Delta f = 0$ everywhere. For suppose $\Delta f(\mathbf{p}) > 0$ for some $\mathbf{p}$ in D. Let B be a ball in D centered at $\mathbf{p}$ in which $\Delta f > 0$, and let ρ be a C^2 function such that $\rho(\mathbf{p}) = 1$ and $\rho = 0$ off B. Then $\rho \in M_0$, so

$$\int_D \rho\Delta f\, dV = \int_B \rho\Delta f\, dV = 0$$

Since $\rho\Delta f \geq 0$ in B, it must be zero. Thus $\Delta f(\mathbf{p}) = \rho(\mathbf{p})\Delta f(\mathbf{p}) = 0$, a contradiction.

Corollary 2. *The orthogonal complement of M_0 in $C^2(D)$ with the inner product $E\langle f, g\rangle$ is the space H of harmonic functions.*

Theorem 8.6. (Dirichlet's Principle) *Let D be a regular domain in R^3 and suppose f is a continuous function on ∂D. Let M_f be the class of functions in $C^2(D)$ with boundary value f.*

(i) *If there is a harmonic function in M_f, it minimizes the energy integral.*

(ii) *If there is a function in $C^2(D)$ which minimizes the energy integral, it must be harmonic.*

Proof. These facts follow from the same reasoning as in Euclidean geometry.

(i) Let $u \in M_f$ be such that $\Delta u = 0$. If g is another function in M_f, $g - u = 0$ on ∂D, so $g - u \in M_0$.

$$E^2\langle g\rangle = E^2\langle g - u + u\rangle = E^2\langle g - u\rangle + 2E\langle g - u, u\rangle + E^2\langle u\rangle$$

$$= E^2\langle g - u\rangle + E^2\langle u\rangle$$

since $u \perp M_0$. Thus, $E^2\langle g\rangle \geq E^2\langle u\rangle$ for every $g \in M_f$.

(ii) If $u \in C^2(D)$ minimizes the energy integral in M_f, it must be orthogonal to M_0. For if $g \in M_0$, then $u \pm g$ are both in M_f, and thus $E^2\langle u + g\rangle \geq E^2\langle u\rangle$. But

$$E^2\langle u \pm g\rangle = E^2\langle u\rangle \pm 2E\langle u, g\rangle + E^2\langle g\rangle$$

so $0 \geq \pm 2E\langle u, g\rangle + E^2\langle g\rangle$ for all $g \in M_0$. Consider for $t \in R$ the function

$$\phi(t) = 2E\langle u, tg\rangle + E^2\langle tg\rangle$$

Since $\phi(t) \leq 0$ for all t (positive or negative), and $\phi(0) = 0$, we must have $\phi'(0) = 0$. But $\phi'(0) = 2E\langle u, g\rangle$. Thus $u \perp M_0$, so, by Corollary 1, u is harmonic.

In order to solve Dirichlet's problem by his principle it remains to show that *there exists* a function in $C^2(D)$ which minimizes the energy integral. The technique for carrying this through was finally accomplished by Hermann Weyl (1926) and his methods have had far reaching effect in a wide class of boundary value problems for partial differential equations.

Harmonic Functions

We can use Green's identities and Dirichlet's principle in order to derive the basic properties of harmonic functions (analogous to those in two dimensions given in Chapter 6). Out of this will come a hint for solving the Dirichlet problem.

Proposition 9. *Let f be a C^2 function defined on the boundary of a regular domain D. There is at most one harmonic function with boundary value f.*

Proof. If u, v are both harmonic and have the boundary values f, then $u - v$ is at the same time harmonic and in M_0. Thus $E\langle u - v, u - v\rangle = 0$. But

$$E\langle u - v, u - v\rangle = \int_D \|\nabla(u - v\|^2 \, dV$$

so we must have $\nabla(u - v) = 0$ in D. Thus $u - v$ is constant. Since $u = v$ on ∂D, u is identical to v.

The gravitational field of a particle of unit mass situated at the point $\mathbf{p}$ is, according to Newton, given as

$$\frac{-1}{\|\mathbf{x} - \mathbf{p}\|^2} \frac{\mathbf{x} - \mathbf{p}}{\|\mathbf{x} - \mathbf{p}\|} \tag{8.58}$$

This field is easily seen to be conservative and divergence free, thus it is the gradient of a harmonic function, called **Newton's gravitational potential.** Writing (8.58) out in coordinates, we have

$$-\frac{(x^1 - p^1, x^2 - p^2, x^3 - p^3)}{[(x^1 - p^1)^2 + (x^2 - p^2)^2 + (x^3 - p^3)^2]^{3/2}}$$

and it is not hard to see that this is the gradient of

$$\Pi_p(\mathbf{x}) = \|\mathbf{x} - \mathbf{p}\|^{-1} = [(x^1 - p^1)^2 + (x^2 - p^2)^2 + (x^3 - p^3)^2]^{-1/2}$$

This particular function stands at the beginning of a sequence of ideas which lead to a technique due to Green, for solving Dirichlet's problem. These steps were motivated by an inquiry into the nature of gravitational fields (due to masses more general than that of a particle), the point being to show that every harmonic function arises as the potential of a gravitational field. Green's first result is an easy consequence (reminiscent of the Cauchy integral formula) of his identities.

Proposition 10. *Let D be a regular domain, and h a function harmonic on D. Let $\mathbf{p} \in D$. Then*

$$h(\mathbf{p}) = \frac{-1}{4\pi} \int_{\partial D} \left[h \frac{\partial \Pi_p}{\partial N} - \Pi_p \frac{\partial h}{\partial N} \right] dS \tag{8.59}$$

Proof. Once again we first remove a small ball $B(\mathbf{p}, \varepsilon)$ centered at $\mathbf{p}$ and contained in D. Since both h, Π_p are harmonic in $D - B(\mathbf{p}, \varepsilon)$, Corollary 1 (iii) applies in that domain. Thus,

$$\int_{\partial[D-B(p,\varepsilon)]}\left[h\frac{\partial\Pi_p}{\partial N}-\Pi_p\frac{\partial h}{\partial N}\right]dS=0$$

This implies that

$$\int_{\partial D}\left[h\frac{\partial\Pi_p}{\partial N}-\Pi_p\frac{\partial h}{\partial N}\right]dS=\int_{\partial B(p,\varepsilon)}\left[h\frac{\partial\Pi_p}{\partial N}-\Pi_p\frac{\partial h}{\partial N}\right]dS \tag{8.60}$$

Now the second integral can be computed using spherical coordinates centered at $\mathbf{p}$:

$$\mathbf{x}=\mathbf{p}+(r\cos\theta\cos\phi,\ r\sin\theta\cos\phi,\ r\sin\theta)$$

Then $\Pi_p(\mathbf{x})=r^{-1}$. The sphere $B(\mathbf{p},\varepsilon)$ is given by

$$\mathbf{x}=\mathbf{p}+\varepsilon(\cos\theta\cos\phi,\ \sin\theta\cos\phi,\ \sin\theta)$$

and its exterior normal is the radial vector, so $\partial/\partial N=\partial/\partial r$. The element of area on $B(\mathbf{p},\varepsilon)$ is $dS=\varepsilon^2\cos^2\phi\,d\theta\,d\phi$. Thus the right-hand side of (8.60) is

$$\begin{aligned}&\int_{\partial B(p,\varepsilon)}\left[h\frac{\partial}{\partial r}\left(\frac{1}{r}\right)-\frac{1}{r}\frac{\partial h}{\partial r}\right]dS\\ &=\int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2}\left[-\frac{h(x)}{\varepsilon^2}-\frac{1}{\varepsilon}\frac{\partial h}{\partial r}\right]\varepsilon^2\cos^2\phi\,d\theta\,d\phi\\ &=-\int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2}h(\mathbf{x})\cos^2\phi\,d\theta\,d\phi-\varepsilon\int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2}\frac{\partial h}{\partial r}\cos^2\phi\,d\theta\,d\phi\end{aligned}$$

Since $|\partial h/\partial r|\le\|\nabla h\|$, the second integrand is bounded as $\varepsilon\to 0$. Thus the second term will vanish for $\varepsilon\to 0$. As for the first term $\mathbf{x}\to\mathbf{p}$ as $\varepsilon\to 0$, so $h(\mathbf{x})\to h(\mathbf{p})$. Thus, letting $\varepsilon\to 0$ our integral tends to

$$-h(\mathbf{p})\int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2}\cos^2\phi\,d\theta\,d\phi=-4\pi h(\mathbf{p})$$

which is what was desired.

Now, if D is the ball of radius R centered at $\mathbf{p}$, then $\Pi_p(\mathbf{x})=\|\mathbf{x}-\mathbf{p}\|^{-1}$, so on D, $\Pi_p=R^{-1}$ and $\partial\Pi_p/\partial N=-R^{-2}$. Equation (8.59) becomes

$$h(\mathbf{p})=\frac{1}{4\pi R^2}\int_{\|x-p\|=R}h\,dS+\frac{1}{4\pi R}\int_{\|x-p\|=R}\frac{\partial h}{\partial N}\,dS$$

Since h is harmonic in D the second integral vanishes (Problem 47) and we obtain the mean value property for harmonic functions in three variables.

Proposition 11. **(Gauss' Theorem)** *If h is harmonic in a neighborhood of $B(\mathbf{p}, R)$, then h satisfies the mean value property:*

$$h(\mathbf{p}) = \frac{1}{4\pi R^2} \int_{\|x-p\|=R} h\, dS$$

Green's Function

Now, by Corollary 1(iii), if k is any function harmonic on D, then (8.59) can be modified by k:

$$h(\mathbf{p}) = \frac{-1}{4\pi} \int_{\partial D} \left[h \frac{\partial(\Pi_p - k)}{\partial N} - (\Pi_p - k) \frac{\partial h}{\partial N} \right] dS \tag{8.61}$$

Thus, if k is chosen so as to solve Dirichlet's problem with the boundary value Π_p, the second term will vanish and we obtain an integral formula for h in terms only of its boundary values. Finally, we could use that formula to solve Dirichlet's problem with any boundary values. Thus (8.59) allows us to reduce the general problem to that for a certain family $\{\Pi_p\}$ of specific functions, and for many regular domains that solution is easily found.

Definition 13. Let D be a domain in R^3. If k_p solves Dirichlet's problem with the boundary values Π_p, we shall call the function $G_p = k_p - \Pi_p$ the **Green's function with singularity at** p.

Theorem 8.7. *Suppose D is a regular domain such that there is a Green's function for every point $\mathbf{p}$ in D. Then if h is harmonic on D, h can be found in terms of its boundary values:*

$$h(\mathbf{p}) = \frac{1}{4\pi} \int_{\partial D} h \frac{\partial G_p}{\partial N}\, dS$$

Proof. By (8.61),

$$h(\mathbf{p}) = \frac{1}{4\pi} \int_{\partial D} \left[h \frac{\partial}{\partial N}(k_p - \Pi_p) - (k_p - \Pi_p) \frac{\partial h}{\partial N} \right] dS$$

but the second integral vanishes since $k_p - \Pi_p = 0$ on ∂D.

Example

35. Let us take D to be the upper half space $D = \{(x, y, z): z \geq 0\}$. Then $\partial D = \{(x, y, 0): (x, y) \in R^2\}$. Since the domain is infinite we have to restrict attention to functions for which the integrals make sense. If H_t is a large hemisphere:

$$H_t = \{(x, y, z): x^2 + y^2 + z^2 \leq t, z \geq 0\}$$

then (8.61) holds for functions h harmonic on D:

$$\begin{aligned} h(\mathbf{p}) = \frac{-1}{4\pi} \int_{\substack{\partial B(0, t) \\ z \geq 0}} &\left[h \frac{\partial \Pi_p}{\partial N} - \Pi_p \frac{\partial h}{\partial N} \right] dS \\ + \frac{1}{4\pi} \int_{\substack{\{z=0\} \\ \{x^2+y^2 \leq t^2\}}} &\left[h \frac{\partial \Pi_p}{\partial N} - \Pi_p \frac{\partial h}{\partial N} \right] dS \end{aligned} \tag{8.62}$$

We shall call the function h **dissipative** if the first integral tends to 0 as $t \to \infty$, and the second integral converges. For example, if $\|\mathbf{x}\|^2 h(\mathbf{x})$ and $\|\mathbf{x}\|^2 \nabla h(\mathbf{x})$ are bounded functions on D, h is dissipative (Problem 48). This is true for Π_p, $\mathbf{p}$ not the on xy plane. Now if h is dissipative we can let $t \to \infty$ in (8.62) and obtain

$$h(\mathbf{p}) = \frac{1}{4\pi} \int_{\{z=0\}} \left[h \frac{\partial \Pi_p}{\partial N} - \Pi_p \frac{\partial h}{\partial N} \right] dS$$

Now if $\mathbf{p} = (x_0, y_0, z_0)$,

$$\Pi_p(\mathbf{x}) = [(x - x_0)^2 + (y - y_0)^2 + (z - z_0)]^{-1/2}$$

and its boundary values ($z = 0$) are the same as those for Π_q where $\mathbf{q} = (x_0, y_0 - z_0)$. Since Π_q is harmonic in D and dissipative, there is a Green's function. Thus, the Green's function for $\mathbf{p} = (x_0, y_0, z_0)$ is

$$\begin{aligned} G_p(\mathbf{x}) &= \Pi_q(x) - \Pi_p(\mathbf{x}) \\ &= \frac{1}{[(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2]^{1/2}} \\ &\quad - \frac{1}{[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2}} \end{aligned}$$

Now the exterior normal to the plane is the downward vertical, so $\partial/\partial N = \partial/\partial z$. A final computation gives

$$\frac{\partial G_p}{\partial N}(\mathbf{x}) = -\frac{\partial G_p}{\partial z}(x, y, 0) = \frac{2z_0}{[(x-x_0)^2 + (y-y_0)^2 + z_0^2]^{3/2}}$$

Thus, if h is harmonic and dissipative in the upper half space, we have for any $z_0 > 0$

$$h(x_0, y_0, z_0) = \frac{z_0}{2\pi}\int_0^\infty \int_0^\infty \frac{h(x, y, 0)}{[(x-x_0)^2 + (y-y_0)^2 + z_0^2]^{3/2}}\, dx\, dy \tag{8.63}$$

Finally, we remark that (8.59) can be used to solve Neumann's problem in the same sense. If there is a harmonic function k_p for each $\mathbf{p}$ in D such that

$$\frac{\partial k_p}{\partial N} = \frac{\partial \Pi_p}{\partial N} \quad \text{on} \quad \partial D$$

then for any function h harmonic on D we have

$$h(\mathbf{p}) = \frac{1}{4\pi}\int_{\partial D} (\Pi_p - k_p)\frac{\partial h}{\partial N}\, dS$$

Thus h is determined by its normal derivative on the boundary.

• PROBLEMS

39. Prove Corollary 2 of Theorem 8.5.

Green's Function for a Ball

40. Using a little bit of plane geometry it is possible to discover the Green's function for the unit ball. If $\mathbf{P}$ is a point inside the ball, let $\mathbf{Q}$ be the point inverse to $\mathbf{P}$ in the sphere

$$\mathbf{Q} = \frac{\mathbf{P}}{\|\mathbf{P}\|^2}$$

Now let **X** be a point on the sphere. Verify that the triangles (see Figure 8.17) **0PX** and **0XQ** are similar (since the angles **P0X** and **Q0X** are the same and

$$|\mathbf{Q0}| = \frac{1}{|\mathbf{P0}|} \quad \text{or} \quad \left|\frac{\mathbf{Q0}}{\mathbf{0X}}\right| = \left|\frac{\mathbf{0X}}{\mathbf{P0}}\right|$$

Conclude that

$$\left|\frac{\mathbf{QX}}{\mathbf{PX}}\right| = \left|\frac{\mathbf{0Q}}{\mathbf{0X}}\right|$$

41. From the above problem we deduce that

$$\Pi_p(\mathbf{x}) = \frac{1}{\|\mathbf{p}\|}\,\Pi_q(\mathbf{x})$$

where $\mathbf{q}$ is the point inverse to $\mathbf{p}$ in the unit sphere. Since $\Pi_q(\mathbf{x})$ is harmonic in the unit ball B, the Green's function for B is

$$\begin{aligned} G_p(\mathbf{x}) &= \frac{\Pi_q(\mathbf{x})}{\|\mathbf{p}\|} - \Pi_p(\mathbf{x}) \\ &= \frac{\|\mathbf{p}\|}{\|\mathbf{p} - \mathbf{x}\|\mathbf{p}\|^2\|} - \frac{1}{\|\mathbf{p} - \mathbf{x}\|} \end{aligned}$$

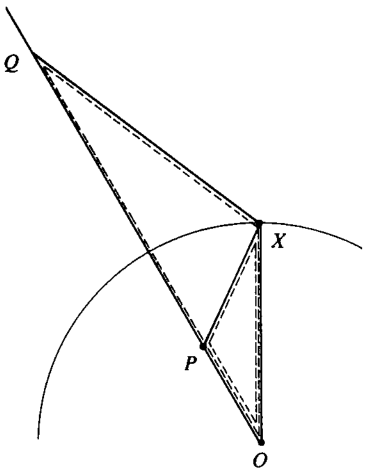

Figure 8.17

Calculate the precise form of Theorem 8.7 (known as **Poisson's formula for the ball)** if h is harmonic on the unit ball

$$h(\mathbf{p}) = \frac{1}{4\pi} \int_{\|\mathbf{x}\|=1} h(\mathbf{x}) \frac{1 - \|\mathbf{p}\|^2}{\|\mathbf{p}\|(\|\mathbf{p} - \mathbf{x}\|)^3} \, dS$$

42. Solve Dirichlet's problem for the ball.
43. Solve Neumann's problem for the ball.
44. Find the steady state temperature distribution in the ball if the surface temperature on the sphere is maintained at
   (a) $\cos \phi$, ϕ is the angle between the point and the north pole.
   (b) $A(x + 2y)$, A a constant.
   (c) $x^2 + y^2 - 2z^2$.
   (d) $\cos 4\theta \sin 2\phi$, θ, ϕ spherical coordinates.
45. Suppose D is a domain for which there exists a Green's function G_p for all $\mathbf{p} \in D$. Show that if $\mathbf{p} \neq \mathbf{p}'$

$$G_p(\mathbf{p}') = G_{p'}(\mathbf{p})$$

(*Hint*: Show, by Green's identity that the integral

$$0 = \left[G_p \frac{\partial G_{p'}}{\partial N} - G_{p'} \frac{\partial G_p}{\partial N} \right] dS$$

is the same as

$$\int_{\partial B - \partial B'} \left[G_p \frac{\partial G_{p'}}{\partial N} - G_{p'} \frac{\partial G_p}{\partial N} \right] dS$$

where B, B' are balls of radius ε centered at $\mathbf{p}$, $\mathbf{p}'$, respectively. Now, using the fact that

$$G_p = \frac{1}{r} + \text{harmonic}$$

compute the limits as $\mathbf{p} \to \mathbf{0}$.)

46. Suppose D, D' are domains with Green's functions G_D, $G_{D'}$ and $D \supset D'$. Show that for $\mathbf{p} \in D'$

$$G_{D,p}(\mathbf{x}) \geq G_{D',p}(\mathbf{x}) \qquad \text{all } \mathbf{x} \text{ in } D'$$

47. Show that for h harmonic in the ball of radius R centered at $\mathbf{p}$,

$$\int_{\|x-p\|=R} \frac{\partial h}{\partial N} \, dS = 0$$

48. Show that the function h defined on the upper half space $D = \{z \geq 0\}$ is dissipative if

$$\|\mathbf{x}\|^2 h(\mathbf{x}) \qquad \|\mathbf{x}\|^2 \nabla h(\mathbf{x})$$

are bounded.

49. Show that if h is harmonic and dissipative in the upper half space and zero on the $z = 0$ plane, then h is identically zero.

50. Suppose that $h(x, y)$ is dissipative on the plane. Prove that there exists a unique dissipative function u continuous on the upper half space $\{z \geq 0\}$ and harmonic for $\{z > 0\}$ which attains the boundary values h. u is given by

$$u(x, y, z) = \frac{z_0}{2} \int_0^\infty \int_0^\infty \frac{h(x, y)}{[(x - x_0)^2 + (y - y_0)^2 + z_0{}^2]^{3/2}} \, dx \, dy$$

51. Find the steady state dissipative temperature distribution on the upper half plane if the temperature on the plane $z = 0$ is maintained at $\exp(x^2 + y^2)^{-1}$.

8.7 Summary

A *fluid flow* is given by a C^1 R^3-valued function $\boldsymbol{\phi}(\mathbf{x}_0, t)$ defined for $\mathbf{x}_0$ in some domain D in R^3 and t on an interval in R about the origin. $\boldsymbol{\phi}$ has these properties:

(i) $\boldsymbol{\phi}(\mathbf{x}_0, 0) = \mathbf{x}_0 \qquad$ all $\mathbf{x}_0 \in D$.

(ii) For fixed t, $\mathbf{x}_0 \to \boldsymbol{\phi}(\mathbf{x}_0, t)$ is one-to-one and has a nonsingular differential.

The vector field

$$\mathbf{v}(\mathbf{x}, t) = \left. \frac{\partial \boldsymbol{\phi}(\mathbf{x}_0, t)}{\partial t} \right|_{x_0 = \phi^{-1}(x, t)}$$

is the *velocity field* of the flow. The flow is *steady* if $\mathbf{v}$ is independent of t.

If $\mathbf{v} = (v_1, v_2, v_3)$ is a differentiable vector field, its *divergence* is the function

$$\operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x^1} + \frac{\partial v_2}{\partial x^2} + \frac{\partial v_3}{\partial x^3}$$

EQUATION OF CONTINUITY. If $\mathbf{v}(\mathbf{x}, t)$ is the velocity field of a flow and $\rho(\mathbf{x}, t)$ is its density, the law of conservation of mass implies

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = \frac{\partial \rho}{\partial t} + \sum v_i \frac{\partial \rho}{\partial x^i} + \rho \operatorname{div} \mathbf{v} = 0$$

A flow is *incompressible* if the same mass always occupies the same volume. The necessary and sufficient condition for this is $\operatorname{div} \mathbf{v} = 0$, where $\mathbf{v}$ is the velocity field of the flow. The fluid is incompressible if and only if the *density at a particle* is constant under all flows of the fluid.

INTEGRATION UNDER A COORDINATE CHANGE. Let $(u, v, w) = \mathbf{F}(x, y, z)$ be a change of coordinates taking a domain D onto the domain Δ. If g is continuous on D,

$$\int_D g(x, y, z)\, dx\, dy\, dz = \int_\Delta g(\mathbf{F}^{-1}(u, v, w)) \left| \det \frac{(x, y, z)}{(u, v, w)} \right| du\, dv\, dw$$

If $\mathbf{v}$ is the velocity field of a flow, the *circulation around* a curve C is defined as

$$\operatorname{circ}(C) = \int_C \langle \mathbf{v}, \mathbf{T} \rangle\, ds$$

If we fix the point $\mathbf{x}_0$ and vector $\mathbf{n}$ at $\mathbf{x}_0$ let C_r be the circle in the plane perpendicular to $\mathbf{n}$ of radius r centered at x_0. The *curl of the flow about* $\mathbf{n}$ *at* $\mathbf{x}_0$ is

$$\operatorname{curl} \mathbf{v}(\mathbf{x}_0, \mathbf{n}) = \lim_{r \to 0} \frac{\operatorname{circ}(C_r)}{r^2}$$

If $\mathbf{v} = (v^1, v^2, v^3)$ define

$$\operatorname{curl} \mathbf{v} = \left(\frac{\partial v^2}{\partial x^3} - \frac{\partial v^3}{\partial x^2}, \frac{\partial v^3}{\partial x^1} - \frac{\partial v^1}{\partial x^3}, \frac{\partial v^1}{\partial x^2} - \frac{\partial v^2}{\partial x^1} \right)$$

Then $\operatorname{curl} \mathbf{v}(\mathbf{x}_0, \mathbf{n}) = \langle \operatorname{curl} \mathbf{v}(\mathbf{x}_0), \mathbf{n} \rangle$. A flow is *irrotational* if $\operatorname{curl} \mathbf{v} = 0$.

A *surface patch* in R^3 is the image of a domain D in R^2 under a C^1 map $\mathbf{x} = \mathbf{x}(u, v)$ with these properties:

(i) $\mathbf{x}$ is one-to-one

(ii) the vectors $\mathbf{x}_u = \partial \mathbf{x}/\partial u$, $\mathbf{x}_v = \partial \mathbf{x}/\partial v$ are independent. (u, v) are called *parameters* or *coordinates* for the surface patch. A surface is a set Σ in R^3 which can be covered by surface patches. The *tangent plane* to Σ is the plane spanned by the vectors $\mathbf{x}_u, \mathbf{x}_v$ (this is independent of the particular coordinates). The normal $\mathbf{N}$ to a surface is a unit vector defined for each point and orthogonal to the tangent plane there.

The form

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

defined on a surface Σ with coordinates (u, v) by

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle \qquad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle \qquad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle$$

is the *first fundamental form* of the surface. The parametric curves are orthogonal if $F = 0$. The length of a curve on Σ given by $u = u(t)$, $v = v(t)$ is

$$\int ds = \int \left[E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2 \right]^{1/2} dt$$

A *geodesic* is a curve of minimal length. If γ is a geodesic on Σ, then at any point $\mathbf{p}$ on γ the normal to γ is orthogonal to the tangent plane of Σ.

The area of a domain D on a surface Σ is defined by

$$\int_D dS = \int_D \|\mathbf{x}_u \times \mathbf{x}_v\|\, du\, dv$$

The integral of a continuous function f defined on D is

$$\int_D f\, dS = \int_D f \|\mathbf{x}_u \times \mathbf{x}_v\|\, du\, dv$$

These definitions are independent of the parameters chosen.

If Σ is an oriented surface and v is a vector field defined around Σ, the flux of v across Σ is

$$\int \langle \mathbf{v}, \mathbf{N} \rangle\, dS$$

STOKES' THEOREM. If $\mathbf{v}$ is a C^1 vector field defined in a domain U, and Σ is an oriented surface in U and D is a regular domain on Σ, then

$$\int_{\partial D} \langle \mathbf{v}, \mathbf{T} \rangle \, ds = \int_D \langle \operatorname{curl} \mathbf{v}, \mathbf{N} \rangle \, dS$$

DIVERGENCE THEOREM. If $\mathbf{v}$ is a C^1 vector field defined in a neighborhood of a regular domain D in R^3 then, with the exterior normal orientation on ∂D,

$$\int_{\partial D} \langle \mathbf{v}, \mathbf{N} \rangle \, dS = \int_D \operatorname{div} \mathbf{v} \, dV$$

GREEN'S IDENTITIES. Let f, g be two C^2 functions defined on a regular domain D. Then

$$\int_{\partial D} f \frac{\partial q}{\partial N} \, dS = \int_D [f \, \Delta g + \langle \nabla f, \nabla g \rangle] \, dV$$

DIRICHLET'S PRINCIPLE. Let D be a regular domain in R^3 and suppose f is a continuous function on D. Let M_f be the class of C^2 functions on D with boundary values given by f.

(i) If there is a harmonic function in M_f, it minimizes the energy integral

$$E^2(u) = \int \|\nabla u\|^2 \, dV$$

(ii) If there is a C^2 function which minimizes the energy integral, it must be harmonic.

• FURTHER READING

In order to continue the study of the divergence theorem and further related topics one must turn to the notations and ideas of differential forms. The small book

M. Spivak, *Calculus on Manifolds*, W. A. Benjamin, Inc., New York, 1965, gives a clear and direct account of this subject. The book

H. K. Nickerson, D. C. Spencer, and N. Steenrod, *Advanced Calculus*, D. Van Nostrand Company, New York, 1957, was the first to give a complete account of this subject on an advanced calculus level. For a more recent account, with a chapter on potential theory in R^n, see

L. Loomis and S. Sternberg, *Advanced Calculus*, Addison-Wesley, Reading, Mass., 1968.

Other references are

M. E. Munroe, *Modern Multidimensional Calculus*, Addison-Wesley, Reading, Mass., 1963.

E. Butkov, *Mathematical Physics*, Addison-Wesley, Reading, Mass., 1968.

For further study of differential geometry we recommend

S. Struik, *Lectures on Classical Differential Geometry*, Addison-Wesley, Reading, Mass., 1950.

H. Guggenheimer, *Differential Geometry*, McGraw-Hill, NewYork, N.Y., 1963.

• MISCELLANEOUS PROBLEMS

52. Suppose that F is a C^1 function defined in a neighborhood of $\mathbf{p}_0$ in R^3 such that $F(\mathbf{p}_0) = 0$ and $dF(\mathbf{p}_0) \neq 0$. Show that the set $\Sigma = \{p: F(\mathbf{p}) = 0\}$ is a surface patch in some neighborhood of $\mathbf{p}_0$. (*Hint*: Choose coordinates x, y, z so that the forms $dF(\mathbf{p}_0)$, $dx(\mathbf{p}_0)$, $dy(\mathbf{p}_0)$ are independent. Then the transformation $\tilde{\mathbf{F}}(\mathbf{p}) = (x(\mathbf{p}), y(\mathbf{p}), F(\mathbf{p}))$ is invertible. If G is the inverse to $\tilde{\mathbf{F}}$, the function

$$\phi(u, v) = G(u, v, 0)$$

parametrizes Σ.)

53. A family of surfaces in a domain D in R^3 is given implicitly by the equation

$$F(p) = c \tag{8.64}$$

where F is C^1 in D and $dF(\mathbf{p}) \neq 0$. For each c, the set (8.64) determines a surface. Show that the vector field ∇F is the velocity field of a flow whose path lines intersect each surface orthogonally.

54. Find the family of curves which are orthogonal to these families of surfaces:

(a) $x^2 + 2y^2 + z^2 = c$. (c) $x^2 + y^2 = c(z + c)$.
(b) $z^2x^2 = c^2$. (d) $z = c \cos y$.

55. Given a family F of curves in space, there may not exist a family of surfaces orthogonal to F. If say, $\mathbf{v}$ is a vector field tangent to the family F and $\{F(p) = c\}$ is the family of orthogonal surfaces, show that ∇F must be collinear with $\mathbf{v}$. The condition that $\mathbf{v}$ must be collinear with a gradient must be satisfied in order for the path lines associated to $\mathbf{v}$ to have an orthogonal family of surfaces. Show that this condition may be written $\langle \operatorname{curl} \mathbf{v}, \mathbf{v} \rangle = 0$.

56. Show that the family of path lines of the helical flow

$$(x, y, z) = (x_0 \cos t + y_0 \sin t, -x_0 \sin t + y_0 \cos t, z_0 + t)$$

does not admit an orthogonal family of surfaces.

57. Show that if the vector field $\mathbf{v}$ is conservative the family of surfaces $\{\Pi(\mathbf{p}) = c\}$, where Π is a potential for $\mathbf{v}$ is orthogonal to the path lines.

58. Show that, although the vector field

$$\mathbf{v}(x, y, z) = (yx, y, 0)$$

is not conservative, the path lines of its associated flow does admit a family of orthogonal surfaces.

59. Suppose that D is a star-shaped domain in R^3 centered at the origin. That is, if $\mathbf{p} \in D$, then so is the line segment joining $\mathbf{0}$ to $\mathbf{p}$ in D. Suppose that $\mathbf{v}$ is a C^1 vector field defined on D such that $\operatorname{div} \mathbf{v} = 0$. Define the vector field $\mathbf{u}$ by

$$\mathbf{u}(\mathbf{p}) = \int_0^1 [\mathbf{v}(t\mathbf{p}) \times t\mathbf{p}]\, dt$$

Show that $\operatorname{curl} \mathbf{u} = \mathbf{v}$. (*Hint*: Recall Poincaré's lemma (see Theorem 7.5); this is just a generalization. Differentiate under the integral sign, use the condition $\operatorname{div} \mathbf{v} = 0$ and then integrate by parts.)

60. Suppose that $\mathbf{u}$ is a C^1 vector field defined in a neighborhood of a sphere S. Show that

$$\int_S \langle \operatorname{curl} \mathbf{u}, \mathbf{N} \rangle \, dS = 0$$

(Use Stokes' theorem one hemisphere at a time.)

61. Every curl-free vector field defined on $R^3 - \{\mathbf{0}\}$ is a gradient; however there is a divergence-free vector field defined there which is not a curl. For example, take

$$\mathbf{v}_0(\mathbf{p}) = \frac{\mathbf{p}}{\|\mathbf{p}\|^3}$$

Then $\operatorname{div} \mathbf{v} = 0$, but if S is a sphere centered at the origin

$$\int_S \langle \mathbf{v}_0, \mathbf{N} \rangle \, dS = 4\pi$$

so by Problem 60, $\mathbf{v}_0$ is not a curl. It can be shown that if $\mathbf{v}$ is any divergence free field defined in $R^3 - \{0\}$, there is a vector field $\mathbf{u}$ and a constant c such that

$$\mathbf{v} = \operatorname{curl} \mathbf{u} + c\mathbf{v}_0$$

Can you suggest how to define c and $\mathbf{u}$?

Normal Curvature

62. Let Σ be a surface patch in R^3 coordinatized by $\mathbf{x} = \mathbf{x}(u, v)$. Let N be the normal to Σ so chosen that $\mathbf{x}_u \to \mathbf{x}_v \to \mathbf{N}$ is right handed. N can be viewed as a differentiable function of u, v. For p on Σ, $d\mathbf{N}(p)$ is thus an R^3-valued linear map of R^2. By defining

$$d\mathbf{N}(p)(\mathbf{x}_u) = \frac{\partial \mathbf{N}}{\partial u}(p)$$

$$d\mathbf{N}(p)(\mathbf{x}_v) = \frac{\partial \mathbf{N}}{\partial v}(p)$$

we may consider $d\mathbf{N}$ as a mapping of the tangent space $T(\Sigma)_p$ into R^3.

(a) Show that the range of $d\mathbf{N}(\mathbf{p})$ is orthogonal to $\mathbf{N}(\mathbf{p})$. (*Hint*: N is a unit vector.)

(b) Because of (a) $d\mathbf{N}(\mathbf{p})$ can be considered as a linear transformation of $T(\Sigma)_p$ to $T(\Sigma)_p$. Show that $d\mathbf{N}(\mathbf{p})$ is symmetric:

$$\langle d\mathbf{N}(\mathbf{p})\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{v}, d\mathbf{N}(\mathbf{p})\mathbf{w}\rangle$$

(*Hint*: You need only show that

$$\langle d\mathbf{N}(\mathbf{p})(\mathbf{x}_u), \mathbf{x}_v\rangle = \langle \mathbf{x}_u, d\mathbf{N}(\mathbf{p})(\mathbf{x}_v)\rangle)$$

(c) Show that $d\mathbf{N}(\mathbf{p})(\mathbf{v})$ is $\kappa_N(\mathbf{v})$ when $\kappa_N(\mathbf{v})$ is the normal curvature (see Problem 25) of the curve of intersection of the plane through N and $\mathbf{v}$ with Σ.

Since $d\mathbf{N}(\mathbf{p})$ is symmetric on $T(\Sigma)_p$, it has two real eigenvalues and the corresponding eigenspaces are orthogonal. The eigenvalues are called the *principal curvatures* of Σ at $\mathbf{p}$, and the eigendirections are the *principal directions.*

63. The *second fundamental form* on a surface is the form

$$\mathrm{II}(\mathbf{v}) = \langle d\mathbf{N}(\mathbf{p})\mathbf{v}, \mathbf{v}\rangle \qquad \text{for } \mathbf{v} \in T(\Sigma)(\mathbf{p})$$

Show that II can be expressed as

$$\mathrm{II} = L\, du^2 + 2M\, du\, dv + N\, dv^2 \tag{8.65}$$

where

$$L = \left\langle \frac{\partial \mathbf{N}}{\partial u}, \frac{\partial \mathbf{x}}{\partial u} \right\rangle$$

$$M = \left\langle \frac{\partial \mathbf{N}}{\partial u}, \frac{\partial \mathbf{x}}{\partial v} \right\rangle = \left\langle \frac{\partial \mathbf{N}}{\partial v}, \frac{\partial \mathbf{x}}{\partial u} \right\rangle$$

$$N = \left\langle \frac{\partial \mathbf{N}}{\partial v}, \frac{\partial \mathbf{x}}{\partial v} \right\rangle$$

64. Compute the second fundamental form and find the principal directions on these surfaces:

(a) $x^2 + 2y^2 + z^2 = 1$

(b) $2y = x^2$

(c) $x^2 - y^2 = z^2$

(d) $x^2 - y^2 + z^2 = 0$

65. **(Rodriques' Formula)** Show that a curve Γ on a surface Σ is tangent to a principal direction at every point if and only if $d\mathbf{N} + \kappa_N\, d\mathbf{x} = 0$ along Γ. (Such curves are called *lines of curvature.*)

66. Find the lines of curvature on the surface Σ:

(a) Σ is the cylinder given by $\mathbf{x}(u, v) = (\cos u, \sin u, v)$.

(b) Σ is the torus $\mathbf{x}(u, v) = (2 + \cos u)\cos v$, $(2 + \cos u = (2 + \cos u)$ $\cos v$, $(2 + \cos u)\sin v$, $\sin u)$.

(c) Σ is the sphere $x^2 + y^2 + z^2 = 1$.

67. A point $\mathbf{p}$ on a surface Σ is called an *elliptic point* if the principal curvatures have the same sign, a *hyperbolic point* if the principal curvatures have different signs and a *parabolic point* if one principal curvature is zero. Find examples of all three kinds of points on a torus. Show that $\mathbf{p}$ is elliptic, hyperbolic, parabolic as $LN - M^2 > 0$, < 0, $= 0$.

68. Show that at a hyperbolic point in a surface intersects its tangent plane in two curves with zero normal curvature.